Estimating the missing values for the incomplete decision matrix and consistency optimization in emergency management

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abstract

Unconventional emergency decision making not only involves intangible and conflicting criteria, but also needs a fast response to the emergency incident under the cases of time pressure and incomplete information. It might be an effective way to make full use of the outlier data of incident information and skip some direct comparisons between alternatives to make a fast emergency decision. Focusing on the missing judgments estimation issue in an incomplete comparison emergency decision matrix, this paper extends the geometric mean induced bias matrix to estimate the missing judgments and improve the consistency ratios at the same time. The least absolute error method and the least square method are used to optimize the revised geometric mean induced bias matrix and find the missing values. A numerical example with incomplete information is used to demonstrate the proposed models. A case of emergency decision making simulation is also conducted to show how the proposed model is applied in practice. The results show the proposed models are not only capable of completing missing values, but also can efficiently improve the matrix consistency at the same time. In addition, the proposed model can aid emergency managers to make a fast response to unconventional emergency in the case of lacking complete information.

1. Introduction

Over the last few decades, unconventional emergencies increase in frequency and intensity, and unconventional emergency management has increasingly become a hot research topic and an important decision problem in the real world [1–3]. However, it is a difficult task to make an effective unconventional emergency decision making because the multiple influence factors of different and unexpected events usually involve intangible and conflicting criteria, which may lead an unexpected event to different outcomes and evolution directions. Therefore, two of the most popular multi-criteria decision making (MCDM) methods, the analytic hierarchy process (AHP) and analytic network process (ANP), have been extensively applied to the emergency decision making. For example, Levy [4] described the use of ANP to improve flood hazard mitigation in the 1998 Yangtze river floods. Levy and Taji [5] presented a group analytic network process (GANP) to support hazard
planning and emergency management under incomplete information. Ohata et al. [6] integrated AHP and Geographical Information System (GIS) to improve the geographical accessibility of neurosurgical emergency hospitals for elderly people, where pairwise comparison was used to calculate the weights of four criteria, i.e. availability of hospital beds, the maximum road distance of the shortest routes, the elderly population within a 3-km radius and the median road distance of the shortest routes. Manca and Brambilla [7] proposed a methodology based on the AHP to quantitatively assess the emergency preparedness and response in road tunnels. Ju et al. [8] presented a hybrid fuzzy AHP and 2-tuple fuzzy linguistic approach to evaluate emergency response capacity. In the second stage of their model, pairwise comparison technique was used to determine the weights of evaluation criteria and sub-criteria. Ju et al. [9] combined the Dempster–Shafer theory with AHP and TOPSIS to evaluate and select the suitable emergency alternatives.

In the process of emergency management, Cosgrave [10] pointed out that there were three constraints which posed particular problems for emergency managers, i.e. time constraint, limited information on which decisions have to be taken, and decision load constraint from a large number of decisions that emergency managers have to take. In addition, different from traditional decision problems, there are two tough situations the decision makers have to face during the process of making an emergency decision. On the one hand, there are no established rules and principles for the unconventional emergency managers to follow. On the other hand, to effectively control the developing and evolution trend of an unexpected emergency event and reduce the impact caused, emergency decision makers must quickly respond using partial or incomplete information. Therefore, when using AHP/ANP to identify and evaluate the critical influence factors for an unexpected event, or assess the emergency planning, emergency preparedness and emergency response alternatives, decision makers need to conduct \( n(n - 1)/2 \) pairwise comparisons if there are \( n \) criteria/alternatives. It could be possible that the pairwise comparisons are inconsistent or incomplete because of the large number, time pressure, lack of the expertise or incomplete information [12]. Consequently, the inconsistency and the incompleteness issues in a pairwise comparison matrix have been paid more attention to over the past few decades, in which a number of methods and models have been proposed to tackle the inconsistency issue (e.g. [13–23]). For the missing comparisons estimation in an incompleteness comparison matrix, Carmone et al. [24] investigated the effect of reduced sets of pairwise comparisons in the AHP by a Monte Carlo simulation. Hu and Tsai [25] proposed a well-known back propagation multi-layer perception to estimate the missing comparisons of incomplete pairwise comparison matrices in the AHP. Fedrizzi and Giove [26] developed a new method to calculate the missing elements of an incomplete matrix of pairwise comparison values for a decision problem by minimizing a measure of global inconsistency. Gomez-Ruiz et al. [27] developed a model based on the Multi-Layer Perceptron (MLP) neural network to complete missing values and improve the matrix consistency at the same time. Bozóki et al. [28] studied the extension of the pairwise comparison matrix to the case when only partial information is available. Ju [29] proposed a new method to solve multiple criteria group decision making problems with incomplete weight information under linguistic environment. As mentioned previously, to control the evolution of emergency events and reduce the impact of emergency events caused, the emergency decision makers must quickly respond to the emergency event and make a fast decision in a short period of time using partial or incomplete information.

The first objective of this paper is to make full use of the critical influence factors (outlier data) and reduce the numbers of pairwise comparisons judgments so as to quickly respond the emergency incident. To achieve such an objective, on the one hand, the emergency evaluation experts are allowed to fill in the most confident pairwise comparisons and disregard certain pairwise comparisons to deal with the lack of knowledge and incomplete information required to make the judgments. On the other hand, emergency decision makers can ask emergency experts to judge the critical influence factors based on the outlier data collected from different scenarios of happened incidents to save time. In this case, we will obtain a number of incomplete comparison matrices, which leads to the second objective of this paper.

The second objective of this paper is to propose a model to quickly estimate the missing comparisons in an incomplete matrix while keeping its consistency. To achieve such an objective, we extend the geometric mean induced bias matrix (for short GMIBM hereinafter) proposed by Ergu et al. [30] to estimate the missing comparisons in an incomplete comparison emergency decision matrix whilst keeping its global consistency. Different from the revised geometric mean (RGM) method proposed by Gomez-Ruiz et al. [27], the adapted GMIBM only requires the original information of the incomplete comparison matrix and is independent of the way of deriving priority weights from a pairwise matrix. Specifically, the missing judgments are first filled in by unknown variables, and then the adapted GMIBM is applied to obtain a revised ‘complete’ pairwise matrix. To keep the global consistency and estimate the missing judgments, the least absolute error (LAE) method and the least square method (LSM) are used to optimize the objective function and find the optimal solution of missing comparisons.

The main contributions of this paper are fourfold:

1. We extend the GMIBM to estimate the missing judgments and improve the consistency ratio at the same time by making full use of the original incomplete information.
2. We propose to apply the least absolute error method and the least square method to optimize the revised GMIBM and find the missing values.
3. The theorems of two corollaries in Ergu et al. [30] are proved mathematically for the first time.
(4) We propose to skip the unimportant assessment factors and focus on the extremely important factors during the process of unconventional emergency decision making, and apply the proposed method to estimate the missing comparisons.

The rest of this paper is organized as follows: next section briefly describes the theorem of GMIBM. In Section 3, the theorem of GMIBM for incomplete comparison matrix is extended. The least absolute error and the least square method are also introduced to construct the optimization problems and find the optimal solution for the missing comparisons. Subsequently, a numerical example and a practical emergency decision making simulation are used to demonstrate how the proposed model can be applied in practice in Section 4. We conclude this paper in Section 5.

2. The theorem of geometric mean induced bias matrix (GMIBM)

To identify and adjust the inconsistent elements in a pairwise comparison matrix, Ergu et al. [30] proposed a GMIBM model. In this paper, the GMIBM is further extended to estimate the missing judgment in incomplete matrices under such cases that fast emergency decisions are made either by deliberately ignoring some unimportant comparisons or by using the critical influence factors. Note that the missing judgments can also occur because of the time pressure or lack of complete information. Since the revised complete comparison matrices should satisfy the consistency requirement, we first briefly review the related theorems and corollaries of GMIBM for consistency case next.

**Theorem 1.** The GMIBM C should be a U matrix if judgment matrix A is perfectly consistent, that is,

\[ C = A \circ A^T = (c_{ij}) = \left( \prod_{k=1}^{n} a_{ik} a_{kj} \right)^{\frac{1}{n}} = U \text{ if } a_{ik} a_{kj} = a_{ij}, \]

where \( A = (a_{ij})_{n \times n} \) represents an n-by-n geometric mean matrix composed of all geometric mean of \( a_{ik} a_{kj} \) for all \( i, j \) and \( k \), \( U = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix} \) "n" denotes the order of A, \( A^T \) represents the transpose of matrix A. Symbol ‘\( \circ \)' denotes Hadamard product (e.g. \( C = A \circ B \) means \( c_{ij} = a_{ij} b_{ij} \) for all \( i \) and \( j \)).

**Theorem 2.** The GMIBM C should be a U matrix if judgment matrix A is perfectly consistent, that is,

\[ C = LR \circ A^T = (c_{ij}) = \left( \prod_{k=1}^{n} a_{ik} \right)^{\frac{1}{n}} \left( \prod_{k=1}^{n} a_{kj} \right)^{\frac{1}{n}} = U \text{ if } a_{ik} a_{kj} = a_{ij}, \]

where \( L = \left( \sqrt[n]{\prod_{k=1}^{n} a_{ik}} \right)_{1 \times n} \) (\( i = 1, \ldots, n \)), represents an n-by-one column matrix composed of geometric mean of rows in matrix A. \( R = \left( \sqrt[n]{\prod_{k=1}^{n} a_{kj}} \right)_{n \times 1} \) denotes an n-by-one row matrix composed of geometric mean of columns in matrix A.

**Corollary 1.** The GMIBM C should be as close as possible to a U matrix if judgment matrix A is approximately consistent.

**Proof.** If the judgment matrix is approximately consistent, i.e. \( a_{ik} a_{kj} \approx a_{ij} \) for all \( i, j \) and \( k \). By theorem 2, we have,

\[ c_{ij} = \sqrt[n]{\prod_{k=1}^{n} a_{ik}} \sqrt[n]{\prod_{k=1}^{n} a_{kj}} \approx \sqrt[n]{\prod_{k=1}^{n} a_{ik}} a_{kj} = \sqrt[n]{\prod_{k=1}^{n} a_{ik} a_{kj}} = a_{ij} \]

Therefore, the GMIBM C is close to a U matrix if the judgment matrix is approximately consistent. \( \square \)

**Corollary 2.** There must be some inconsistent elements in the GMIBM C deviating far away from one if the judgment matrix A is inconsistent.

**Proof by contradiction.** Assume all entries in matrix C equal to one even if the judgment matrix A is inconsistent, that is, \( a_{ik} a_{kj} \neq a_{ij} \) holds for some \( i, j \) and \( k \), but \( c_{ij} = 1 \) (\( i, j = 1, \ldots, n \)), namely

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\[ c_{ij} = \sqrt[n]{\prod_{k=1}^{n} a_{ik} \cdot \prod_{l=1}^{n} a_{lj}} \cdot a_{ji} = \sqrt[n]{\prod_{k=1}^{n} a_{ik} a_{kj} \cdot a_{ji} = 1} \]

Since \( a_{ji} = 1/a_{ij} \), we have

\[ a_{ij} = \sqrt[n]{\prod_{k=1}^{n} a_{ik} a_{kj} = \sqrt[n]{\prod_{l=1}^{n} a_{il} a_{lj}} \quad (i,j = 1, \ldots n) \]

Similarly, since \( a_{ik} = 1/a_{ki} \), we can obtain

\[ a_{ik} = \sqrt[n]{\prod_{l=1}^{n} a_{il} a_{kl} a_{lj} = \sqrt[n]{\prod_{l=1}^{n} a_{il} a_{lj}}} \quad \text{and} \quad a_{ji} = \sqrt[n]{\prod_{l=1}^{n} a_{il} a_{lj}}. \]

We previously assume all entries in matrix \( C \) equal to one, thus \( c_{ik} = 1 \) and \( c_{ij} = 1 \). Similar to \( a_{ij} \), we can obtain,

\[ a_{ik} = \sqrt[n]{\prod_{l=1}^{n} a_{il} a_{kl} \text{ and } a_{ji} = \sqrt[n]{\prod_{l=1}^{n} a_{il} a_{lj}}}. \]

Therefore, \( a_{il} a_{kj} = a_{ij} \). The result contradicts the previous assumption that \( a_{ij} = a_{ik} a_{kj} \) for some \( j \) and \( k \), indicating any row or column of matrix \( C \) contains at least one non-one entry. □

### 3. GMIBM for incomplete matrix

#### 3.1. Incomplete pairwise comparison matrix

According to the principle of pairwise comparison technique originated by Thurstone [31] and the theorem of AHP proposed by Saaty [32], if there are \( n \) qualitative criteria, all pair-compared results are arranged in a matrix \( A = (a_{ij})_{n \times n} \), where \( a_{ij} > 0, a_{ij} = 1/a_{ji} \) and \( a_{ij} = a_{ik} a_{kj} \) for \( i, j, k = 1, 2, \ldots, n \), and decision makers need to complete \( n(n-1)/2 \) pairwise comparisons. To quantify experts’ judgments, Professor Saaty suggests using the fundamental scale of the absolute numbers 1–9 to represent judgments in the decision matrix \( A \). As such, an incomplete matrix refers to decision makers could not fill in \( n(n-1)/2 \) pairwise comparisons because of time pressure, unwillingness to make a direct comparisons between alternatives or being unsure of some of the comparisons [13], and there are one or more pairs of missing entries in matrix \( A \), denoted as [28].

\[
A = \begin{pmatrix}
1 & a_{12} & \cdots & a_{1n} \\
a_{21} & 1 & a_{23} & \cdots & \times \\
\times & a_{32} & 1 & \cdots & a_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_{n1} & \times & a_{n3} & \cdots & 1
\end{pmatrix}
\]

where \( a_{ij} \) represent the given values of pairwise comparisons, ‘\( \times \)’ denotes the missing comparisons.

Assume there are \( p \) missing comparisons in the upper triangular part of the incomplete matrix \( A \), we can introduce some unknown variables \( x_1, x_2, \ldots, x_p \) to denote the \( p \) missing comparisons, and then calculate and estimate the missing comparisons by mathematics tools and models. Due to the reciprocal property of pairwise comparison matrix, the total number of missing comparisons of the revised ‘complete’ matrix \( A \) is \( 2p \), and can be written as,

\[
A(a_{ij}, x) = A(a_{ij}, x_1, \cdots, x_p) = \begin{pmatrix}
1 & a_{12} & x_1 & \cdots & a_{1n} \\
a_{21} & 1 & a_{23} & \cdots & x_p \\
1/x_1 & a_{32} & 1 & \cdots & a_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_{n1} & 1/x_p & a_{n3} & \cdots & 1
\end{pmatrix}
\]

To estimate the optimal values of the \( p \) unknown variables in matrix \( A \), in the following, the GMIBM model is further extended and adapted.

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3.2. GMIBM for estimating missing comparisons

For an incomplete pairwise comparison matrix A, the missing comparisons should be estimated while keeping the global consistency so as to make a valid decision. Based on the aforementioned Theorems and Corollaries, we can derive the following theorem for estimating the missing comparisons in an incomplete comparison matrix.

**Theorem 3.** The geometric mean induced bias error matrix (GMIBM) ε should be equal (or close) to a zero matrix if judgment matrix A is perfectly (or approximately) consistent, that is,

$$
\varepsilon = A \circ A^T - U = (c_{ij} - 1) = \left\{ \begin{array}{ll}
0 & \text{if } a_{ik}a_{kj} = a_{ij}, \\
\approx 0 & \text{if } a_{ik}a_{kj} \approx a_{ij},
\end{array} \right.
$$

or

$$
\varepsilon = LR \circ A^T - U = (c_{ij} - 1) = \left\{ \begin{array}{ll}
0 & \text{if } a_{ik}a_{kj} = a_{ij}, \\
\approx 0 & \text{if } a_{ik}a_{kj} \approx a_{ij}
\end{array} \right.
$$

where $A^T$ and $\bar{A}$ are the transpose matrix and the average matrix of the revised ‘complete’ matrix $A$ with unknown variables $x_1$ to the 9-point scale proposed by Saaty [32]. One also can minimize the average absolute error to find the solutions of $\varepsilon$.

To find the optimal values of the unknown variables while keeping the global consistency, we can establish an overdetermined system of equations, in which there are more equations than unknowns, i.e. $\varepsilon(x_1, x_2, \ldots, x_p, a_{ij}) = 0, i, j = 1, 2, \ldots, n$.

The specific steps of the missing comparisons estimation include:

**Step 1:** Fill in the missing comparisons with unknown variables $x_1$, $1/x_1$; $x_2$ and $1/x_2$; etc.

**Step 2:** Construct the GMIBM $\varepsilon$ by the following three sub-steps.

- **Step 2.1:** Compute a column vector $L$ and a row vector $R$ (see formula (2)).
- **Step 2.2:** Compute the geometric mean matrix by formula $\bar{A} = L \times R$.
- **Step 2.3:** Compute GMIBM $\varepsilon$.

**Step 3:** Establish an overdetermined system of equations by minimizing all entries in the error matrix $\varepsilon$, i.e. let $\varepsilon(x_1, x_2, \ldots, x_p, a_{ij}) = 0, i, j = 1, 2, \ldots, n$ hold.

**Step 4:** Solve the overdetermined system of equations.

**Step 5:** Test the revised comparison matrix $A$ by replacing the missing comparisons with the estimated values.

For some complicated overdetermined system of equations, especially those that are generated from incomplete matrices with high orders, sometimes it is difficult to find their explicit solutions. In such case, we can construct the following optimization problems instead of the overdetermined system of equations to find optimal solutions of unknown variables. According to the approximated case in formulas (3) and (4), all absolute errors in the error matrix $\varepsilon$ should be minimized in order to keep the global consistency, thus we have the following optimization problem.

$$
\text{Min} \quad \varepsilon(a_{ij}, x) = |e_{ij}| = \left| |e_{ij}(x_1, x_2, \ldots, x_p, a_{ij})| \right|,
$$

s.t. \quad 1/9 \leq x \leq 9.

To create one objective function $f(a_{ij}, x)$, the commonly used least absolute errors (LAE) method is introduced to simplify the above optimization problem, i.e., all bias absolute error entries in error matrix $\varepsilon$ are added together, and we can further obtain the following optimization problem by minimizing the sum of absolute errors (SAE) of error matrix $\varepsilon$.

$$
\text{Min} \quad f(a_{ij}, x) = \sum_{j=1}^{n} \sum_{i=1}^{n} |e_{ij}| = \sum_{j=1}^{n} \sum_{i=1}^{n} |e_{ij}(x, a_{ij})|,
$$

s.t. \quad 1/9 \leq x \leq 9,

where $a_{ij}$ is the given decision judgments, and $x = (x_1, x_2, \ldots, x_p)$ is the vector of missing comparison variables, which is subject to the 9-point scale proposed by Saaty [32]. One also can minimize the average absolute error to find the solutions of unknown variables, namely, define the corresponding optimization problem as,

$$
\text{Min} \quad f(a_{ij}, x) = \frac{1}{n(n-1)} \sum_{j=1}^{n} \sum_{i=1}^{n} |e_{ij}| = \frac{1}{n(n-1)} \sum_{j=1}^{n} \sum_{i=1}^{n} |e_{ij}(x, a_{ij})|,
$$

s.t. \quad 1/9 \leq x \leq 9.

Analogously, we can construct the following optimization problem by minimizing the squares of each error in error matrix $\varepsilon$ to find the optimal solutions.
Min $\varepsilon(a_{ij}, x) = \left(\varepsilon_{ij}^2\right) = \left(\varepsilon_{ij}(x_1, x_2, \ldots, x_p, a_{ij})^2\right)$.

\[ s.t. \quad 1/9 \leq x \leq 9. \tag{8} \]

In practice, the above optimization problem is usually transformed into the following least square optimization problem.

Min $f(a_{ij}, x) = \sum_{j=1}^{n} \sum_{i=1}^{n} (\varepsilon_{ij}^2) = \sum_{j=1}^{n} \sum_{i=1}^{n} \left(\varepsilon_{ij}(x, a_{ij})^2\right)$,

\[ s.t. \quad 1/9 \leq x \leq 9. \tag{9} \]

The corresponding optimization problem of average least square error can be defined as,

Min $f(a_{ij}, x) = \frac{1}{n(n-1)} \sum_{j=1}^{n} \sum_{i=1}^{n} (\varepsilon_{ij}^2) = \frac{1}{n(n-1)} \sum_{j=1}^{n} \sum_{i=1}^{n} \left(\varepsilon_{ij}(x, a_{ij})^2\right)$,

\[ s.t. \quad 1/9 \leq x \leq 9. \tag{10} \]

4. Experimental simulation

In this section, a numerical example is first used to examine the validity and effectiveness of the aforementioned two ways of estimating the missing comparisons. Then a case of unconventional emergency decision is studied to show the application of the proposed method in the real world decision making.

4.1. Numerical example

Suppose there are four emergency response alternatives that need to be quickly evaluated in a natural disaster emergency incident, denoted as $A_1, A_2, A_3,$ and $A_4$, respectively. For the lack of related information, limitation of expertise and time pressure, one expert only filled in the values of $a_{12}, a_{14},$ and $a_{23}$, the comparisons $a_{13}, a_{24},$ and $a_{34}$ are missing, as shown below.

$$A = \begin{bmatrix} 1 & \frac{1}{9} & \times & \frac{1}{5} \\ 9 & 1 & 5 & \times \\ \times & \frac{1}{5} & 1 & \times \\ 5 & \times & \times & 1 \end{bmatrix} \tag{11}$$

In this simple example, the missing values can easily be estimated by using the perfect consistency condition $a_{ij} = a_{ik}a_{kj}$, that is, $a_{13} = a_{12}a_{23} = 5/9, a_{24} = a_{21}a_{14} = 9/5,$ and $a_{34} = a_{32}a_{24} = 9/25$. In the following, we apply the proposed GMIBEM model into this judgment matrix to demonstrate the implementation process and validate the effectiveness of the proposed optimization model. The details are as follows.

Case-1: Method of solving overdetermined system of equations

Step 1: The revised pairwise comparison matrix $A'$ with unknown variables $x_1, x_2, \text{ and } x_3$ is

$$A' = \begin{bmatrix} 1 & \frac{1}{9} & x_1 & \frac{1}{5} \\ 9 & 1 & 5 & x_2 \\ \frac{1}{x_1} & \frac{1}{5} & 1 & x_3 \\ 5 & \frac{1}{x_2} & \frac{1}{x_3} & 1 \end{bmatrix} \tag{12}$$

Step 2: Construct the GMIBEM $\varepsilon$ by following three sub-steps.

Step 2.1: Compute a column vector $L$ and a row vector $R$.

$$L = A' = \begin{bmatrix} 1 & \frac{1}{9} & x_1 & \frac{1}{5} \\ 9 & 1 & 5 & x_2 \\ \frac{1}{x_1} & \frac{1}{5} & 1 & x_3 \\ 5 & \frac{1}{x_2} & \frac{1}{x_3} & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{45} x_1 \\ 4 \frac{1}{45} x_2 \end{bmatrix}$$

$$R = \begin{bmatrix} \frac{4}{45} x_1 \\ \frac{1}{45} x_2 \\ \frac{4}{5} x_3 \end{bmatrix} \begin{bmatrix} \frac{1}{45} x_1 \\ 4 \frac{1}{45} x_2 \end{bmatrix} \begin{bmatrix} \frac{1}{5} x_2 x_3 \end{bmatrix} \tag{13}$$

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where

\[ R = \begin{pmatrix} \sqrt{\frac{41}{11}} & \sqrt{\frac{5}{25}} & \sqrt{\frac{8}{29}} & \sqrt{\frac{14}{25}} \end{pmatrix} \quad \text{and} \quad L = \begin{pmatrix} \sqrt{\frac{15}{55}} & \sqrt{\frac{25}{21}} & \sqrt{\frac{8}{29}} \end{pmatrix}^T \]

Step 2.2: Compute the geometric mean matrix by formula \( \bar{A} = LR \)

\[
\bar{A} = \begin{bmatrix}
        1 & \frac{x_1}{2025x_2} & \frac{x_1^2}{29x_2} & \frac{x_1 x_2}{225} \\
        \frac{225 x_1 x_2}{x_1} & 1 & \frac{225 x_1 x_2}{x_3} & \frac{9 x_1^3 x_2}{x_3} \\
        \frac{5 x_1 x_2}{x_1} & \frac{5 x_1 x_2}{x_2} & 1 & \frac{x_1 x_2^2}{29 x_1} \\
        \frac{225 x_1 x_2}{x_1} & \frac{5 x_1 x_2}{x_2} & \frac{x_1 x_2^2}{29 x_1} & 1
\end{bmatrix}
\]  

(14)

Step 2.3: Compute GMIBEM \( \varepsilon \).

\[
\varepsilon = \begin{bmatrix}
        0 & 9 \frac{x_1}{2025x_2} - 1 & \frac{1}{x_1} \frac{x_1}{29x_2} - 1 & 5 \frac{x_1 x_2}{225} - 1 \\
        \frac{1}{x_1} \frac{x_1}{2025 x_2} - 1 & 9 \frac{x_1}{2025 x_2} - 1 & \frac{1}{x_1} \frac{x_1}{29 x_2} - 1 & \frac{1}{x_1} \frac{x_1}{29 x_2} - 1 \\
        \frac{1}{x_1} \frac{x_1}{2025 x_2} - 1 & \frac{1}{x_1} \frac{x_1}{2025 x_2} - 1 & \frac{1}{x_1} \frac{x_1}{29 x_2} - 1 & \frac{1}{x_1} \frac{x_1}{29 x_2} - 1 \\
        \frac{1}{x_1} \frac{x_1}{2025 x_2} - 1 & \frac{1}{x_1} \frac{x_1}{2025 x_2} - 1 & \frac{1}{x_1} \frac{x_1}{29 x_2} - 1 & \frac{1}{x_1} \frac{x_1}{29 x_2} - 1
\end{bmatrix}
\]  

(15)

Step 3: Establish the overdetermined system of equations by \( \varepsilon_{ij}(x_1, x_2, \ldots, x_p, a_i) = 0, i,j = 1,2,\ldots,n. \)

\[
\begin{align*}
\begin{cases}
\frac{4}{5} \frac{225 x_1 x_2}{x_1} - 1 = 0 \\
\frac{4}{5} \frac{225 x_1 x_2}{x_1} - 1 = 0 \\
\frac{5}{29 x_1} - 1 = 0 \\
\frac{1}{29 x_1} - 1 = 0 \\
\frac{29 x_1}{29 x_1} - 1 = 0
\end{cases}
\end{align*}
\]

(I)  

\[
\begin{align*}
\begin{cases}
\frac{19}{5} \frac{x_1}{2025 x_2} - 1 = 0 \\
\frac{1}{5} \frac{x_1}{29 x_2} - 1 = 0 \\
\frac{4}{225 x_1 x_2}{x_1} - 1 = 0 \\
\frac{4}{225 x_1 x_2}{x_1} - 1 = 0 \\
\frac{4}{225 x_1 x_2}{x_1} - 1 = 0
\end{cases}
\end{align*}
\]

(II)  

(16)

where 'I' represents the system of equations composed of the lower triangular parts of matrix \( \varepsilon \), while 'II' denotes the system of equations composed of the upper triangular parts of matrix \( \varepsilon \).

Step 4: Solve the overdetermined system of equations (I) and (II), we obtain,

\[
x_1 = \frac{5}{9}, \quad x_2 = \frac{9}{5}, \quad x_3 = \frac{9}{25}
\]

The calculated results are the same as the values derived from the perfect consistency condition.

Step 5: Test the revised comparison matrix \( \bar{A} \) by replacing the missing comparisons with the estimated values.

Since the maximum eigenvalue is \( \lambda_{\text{max}} = 4 \), thus \( CR = 0 \). If one wants to use the 9-point scale integer number, then the integer value that is closest to 9-point scale can be selected as the optimal value of unknown variables. For instance, \( x_1 = 5/9 = 0.5556 \approx 1/2, x_2 = 9/5 = 1.8 \approx 2 \) and \( x_3 = 9/25 = 0.36 \approx 1/3 \), replace the unknown variables with these values and test the consistency, the maximum eigenvalue is \( \lambda_{\text{max}} = 4.0055 \), and the corresponding consistency ratio \( CR = 0.0021 \), which is far less than the consistency ratio threshold 0.1.

Case-2: Methods of least absolute error and least square method optimization  
In addition to solving the overdetermined system of equations, we can find the optimal solutions by using optimization formulas (6) and (9). In the following, both optimization methods are applied to the above numerical example. According to the error matrix \( \varepsilon \) and optimization formula (6), we can construct the least absolute error objective function \( f_i(x) \) by summing up all absolute entries of error matrix.
By least absolute error (LAE thereafter) optimization formula (6), the objective function $f_1(x)$ should be minimized to keep the global consistency, and the variables are subject to the 9-point scale, that is, we can construct the following optimization model,

$$\text{Min } f_1(x)$$

s.t. \begin{align*}
1/9 \leq x_1 \leq 9 \\
1/9 \leq x_2 \leq 9 \\
1/9 \leq x_3 \leq 9
\end{align*}

(18)

Apply the nonlinear constrained optimization function fmincon in Matlab software to solve this optimization problem, we can get the final solutions of unknown variables $x, y$ and $z$ and the aggregation values of objective function $f_1(x)$ after 35 iterations of the algorithm, as shown in Table 1. To display the whole iteration steps of optimization, we plotted the objective function’s value and the final estimated values of unknown variables. Fig. 1(b) shows that the decrease of objective function value is close to zero after the 9th iteration, in which the function value in the 9th iteration is 0.0751224 as listed in Table 1.

The least square method (LSM) optimization problem becomes,

$$f_2(x) = \left( \frac{1}{9} \sqrt{\frac{2025x_2}{x_1}} - 1 \right)^2 + \left( x_1 \sqrt{\frac{9x_1}{x_1^2}} - 1 \right)^2 + \left( \frac{1}{5} \sqrt{\frac{225}{x_1x_2x_3}} - 1 \right)^2 + \left( 5 \sqrt{\frac{x_3}{225x_1x_2}} - 1 \right)^2 + \left( x_2 \sqrt{\frac{1}{9x_2^2x_3}} - 1 \right)^2$$

(19)

$$f_2(x) = \left( \frac{1}{9} \sqrt{\frac{2025x_2}{x_1}} - 1 \right)^2 + \left( x_1 \sqrt{\frac{9x_1}{x_1^2}} - 1 \right)^2 + \left( \frac{1}{5} \sqrt{\frac{225}{x_1x_2x_3}} - 1 \right)^2 + \left( 5 \sqrt{\frac{x_3}{225x_1x_2}} - 1 \right)^2 + \left( x_2 \sqrt{\frac{1}{9x_2^2x_3}} - 1 \right)^2$$

The least square method (LSM) optimization problem becomes,
To compare with the LAE method, the optimization function \textit{fmincon} is again applied to solve above optimization problem, the detailed iterations and changes of $f_2(x)$ are also listed in Table 1 and plotted in Fig. 2. Fig. 2(b) shows that the function value decreases drastically after the first iteration, and it almost reaches zero after the second iteration. Optimization stopped in the 9th iteration since the predicted change in the objective function, $5.5128 \times 10^{-8}$, is less than the given threshold options, $10^{-6}$. The values of estimated unknown variables showed in Table 1 and plotted in Fig. 2(a) are almost the same as the results obtained from case 1. Therefore, the revised complete matrix by these estimated values satisfies perfect consistency condition.
4.2. A case of emergency decision making simulation

We have previously demonstrated the implementation of the proposed method by a 4 × 4 incomplete matrix. In the following, we apply the proposed method to a case of emergency decision making with incomplete decision information in the real world. During the process of making decision for unconventional emergency, quick assessment and scenario-response are extremely important to save lives and reduce the property losses.

Take the Yushu earthquake analyzed by Liu et al. [33] as an example, assume an emergency manager has to quickly evaluate the eight effectiveness assessment indicators proposed by Liu et al. [34] for different scenarios using pairwise comparisons technique. They are: Casualties, Personnel rescued, Rescue workers input, Materials input, Direct economic losses, Indirect economic losses, Effectiveness of the physical environment, Social benefits, denoted as C1, C2, C3, C4, C5, C6, C7 and C8, respectively. According to the pairwise comparisons technique, the manager needs to ask the emergency experts to fill in n(n − 1)/2 = 28 numbers of pairwise comparisons. To quickly evaluate these criteria and make a fast decision, the emergency manager decided to reduce the number of comparisons by allowing the surveyed emergency experts to fill in parts of the comparisons, i.e., filling in the most confident comparisons using their experience and expertise knowledge.

Assume the emergency manager collected the following incomplete matrix A with 12 missing comparisons from one of the surveyed experts, where missing comparisons are denoted by ‘×’. To estimate the missing comparisons and evaluate these criteria, we apply the proposed model to this matrix.

\[
A = \begin{bmatrix}
1 & 5 & 3 & 7 & 6 & 6 & 1/3 & 1/4 \\
1/5 & 1 & × & 5 & × & 3 & × & 1/7 \\
1/3 & × & 1 & × & 3 & × & 6 & × \\
1/7 & 1/5 & × & 1 & × & 1/4 & × & 1/8 \\
1/6 & × & 1/3 & × & 1 & × & 1/5 & × \\
1/6 & 1/3 & × & 4 & × & 1 & × & 1/6 \\
3 & × & 1/6 & × & 5 & × & 1 & × \\
4 & 7 & × & 8 & × & 6 & × & 1
\end{bmatrix}
\]

First, the missing comparisons are replaced by twelve unknown variables, \(x_i (i = 1,2,\ldots,12)\), we can obtain the revised ‘complete’ matrix with variables, denoted as \(A(x)\).

\[
A(x) = \begin{bmatrix}
1 & 5 & 3 & 7 & 6 & 6 & 1/3 & 1/4 \\
1/5 & 1 & x_1 & 5 & x_2 & 3 & x_3 & 1/7 \\
1/3 & 1/x_1 & 1 & x_4 & 3 & x_5 & 6 & x_6 \\
1/7 & 1/5 & 1/x_4 & 1 & x_7 & 1/4 & x_8 & 1/8 \\
1/6 & 1/x_2 & 1/3 & 1/x_7 & 1 & x_9 & 1/5 & x_{10} \\
1/6 & 1/3 & 1/x_5 & 4 & 1/x_9 & 1 & x_{11} & 1/6 \\
3 & 1/x_3 & 1/6 & 1/x_6 & 5 & 1/x_{11} & 1 & x_{12} \\
4 & 7 & 1/x_6 & 8 & 1/x_{10} & 6 & 1/x_{12} & 1
\end{bmatrix}
\]

To construct and demonstrate the proposed optimization models, both LAE and LSM are used in the following.

Case 1: least absolute error (LAE) method

According to formula (6) of LAE, the optimization problem is

\[
\begin{align*}
\text{Min} \quad & f(a_{ij}, x) = \sum_{j=1}^{8} \sum_{i=1}^{8} |a_{ij}| = \sum_{j=1}^{8} \sum_{i=1}^{8} \left| \prod_{k=1}^{8} a_{ik}(x) \cdot a_{ji}(x) \cdot a_{ji}(x) \right| - 1, \\
\text{s.t.} \quad & 1/9 \leq x \leq 9.
\end{align*}
\]

Or

\[
\begin{align*}
\text{Min} \quad & f(a_{ij}, x) = \frac{1}{8} \sum_{j=1}^{8} \sum_{i=1}^{8} |a_{ij}| = \frac{1}{56} \sum_{j=1}^{8} \sum_{i=1}^{8} \left| \prod_{k=1}^{8} a_{ik}(x) \cdot a_{ji}(x) \right| - 1, \\
\text{s.t.} \quad & 1/9 \leq x \leq 9.
\end{align*}
\]

where \(a_{ij}(x)\) is the \(i\)th row and \(j\)th column entry of the revised ‘complete’ matrix \(A(x)\), and \(x=(x_1, x_2, \ldots, x_{12})\) is the vector of unknown variables, which is subject to 1/9 to 9 in terms of the 9-point scale proposed by Saaty [32].

Apply the nonlinear constrained optimization function \(fmincon\) in Matlab software to solve the optimization problem (23), the detailed values of objective function of each iteration optimization and the final estimated optimal values of variables by LAE are shown in Table 2 and plotted in Fig. 3.
Fig. 3 shows the changes of objective function’s value and the final estimated values of unknown variables during the whole iteration steps, in which we can observe that a significant decrease of objective function happens in the 8th iteration, as shown in Fig. 3(b). After 24th iteration, the decrease of objective function becomes unobvious, i.e. the decrease only happens at the decimal fraction part of the function value. Finally, the optimization terminated in the 80th iteration. The value of final minimized objective function is 23.3604, and the corresponding average absolute error of each entry by formula (24) is 

\[
\frac{23.3604}{56} = 0.41715.
\]

The optimal estimated values of unknown variables showed in Fig. 3(a) are

\[x_i = (0.3807, 1.8279, 0.4735, 9, 4.2863, 0.5264, 0.5320, 0.1378, 0.8928, 0.1111, 0.2902, 0.4232), i = 1, 2, \ldots, 12.\]

Replace the missing comparisons in matrix \(A(x)\) with the above optimal values, we can get the revised complete matrix \(A(x')\),

\[
A(x') = \begin{bmatrix}
1 & 5 & 3 & 7 & 6 & 6 & 1/3 & 1/4 \\
1/5 & 1 & 0.3807 & 5 & 1.8278 & 3 & 0.4735 & 1/7 \\
1/3 & 1/0.3807 & 1 & 9 & 3 & 4.2864 & 6 & 0.5264 \\
1/7 & 1/5 & 1/9 & 1 & 0.532 & 1/4 & 0.1378 & 1/8 \\
1/6 & 1/1.8278 & 1/3 & 1/0.532 & 1 & 0.8928 & 1/5 & 0.1111 \\
1/6 & 1/3 & 1/4.2864 & 4 & 1/0.8928 & 1 & 0.2902 & 1/6 \\
3 & 1/0.4735 & 1/6 & 1/0.1378 & 5 & 1/0.2902 & 1 & 0.4232 \\
4 & 7 & 1/0.5264 & 8 & 1/0.1111 & 6 & 1/0.4232 & 1
\end{bmatrix}
\]

Table 2
The results of optimization by least absolute error (LAE) method.

<table>
<thead>
<tr>
<th>Iter</th>
<th>(f_2(x))</th>
<th>Iter</th>
<th>(f_3(x))</th>
<th>Iter</th>
<th>(f_4(x))</th>
<th>Iter</th>
<th>(f_5(x))</th>
<th>Iter</th>
<th>(f_6(x))</th>
<th>Variables (x_i)</th>
<th>Estimated values</th>
<th>9-Point scale</th>
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<td>1/4</td>
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</table>

Fig. 3. The values of unknown variables and the changing function value by LAE.
Calculate the maximum eigenvalue of the revised matrix, we can obtain $\lambda_{\text{max}}^{\text{LAE}} = 9.3024$. The corresponding right eigenvector is calculated and listed in Table 4 in order to compare it with the results obtained by LSM. If one wants to use the 9-point integer scale proposed by Saaty, it is recommended to choose the closest value to the 9-point integer scale, as shown in the right column in Table 2, the corresponding maximum eigenvalue of these estimated integer values is $\lambda_{\text{max}}^{\text{LSM}} = 9.3125$.

Case 2: Least square method (LSM)

In addition to LAE method, the LSM could also be used to estimate the optimal values of missing comparisons by minimizing the sum of errors squares of formula (4). The corresponding optimization problem of above example can be constructed as,

$$\min f(a_{ij}, x) = \sum_{j=1}^{8} \sum_{i=1}^{8} (x_{ij} - \sum_{k=1}^{8} a_{ik}(x) a_{kj}(x))^{2},$$

s.t. $1/9 \leq x \leq 9$.

(26)

Or

$$\min f(a_{ij}, x) = \frac{1}{8 \times 7} \sum_{j=1}^{8} \sum_{i=1}^{8} (x_{ij} - \sum_{k=1}^{8} a_{ik}(x) a_{kj}(x))^{2},$$

s.t. $1/9 \leq x \leq 9$.

(27)

Similar to LAE method, we apply the nonlinear constrained optimization function fmincon in Matlab software to solve optimization problem (26), the detailed values of objective function of each iteration optimization and the final estimated optimal values can be obtained, as shown in Table 3, and are plotted in Fig. 4. It can be seen from Fig. 4(b) that a significant decrease of objective function happens in the 5th iteration. After 19th iteration, the decrease of objective function only happens on the decimal fraction part of the function values, and the value of final minimized objective function is 33.8381. By formula (27), the corresponding average least square error is 33.8381/56 = 0.6043. Fig. 4(a) also shows the dominated variables are $x_4$ and $x_5$. It only took 4.818181 s to estimate the 12 missing variables on a 2.5 GHz Pentium laptop. The speed could be improved if high configuration hardware is employed.

Replace the missing comparisons in matrix $A(x)$ with the estimated optimal values showed on the most right column in Table 3, the revised complete matrix $A(x')$ is,

$$A(x') = \begin{bmatrix} 1 & 5 & 3 & 7 & 6 & 6 & 1/3 & 1/4 \\ 1/5 & 1 & 0.2633 & 5 & 1.6536 & 3 & 0.5038 & 1/7 \\ 1/3 & 1/0.2633 & 1 & 9 & 3 & 6.1232 & 6 & 0.7461 \\ 1/7 & 1/5 & 1/9 & 1 & 0.5027 & 1/4 & 0.1531 & 1/8 \\ 1/6 & 1/1.6536 & 1/3 & 1/0.5027 & 1 & 0.9743 & 1/5 & 0.1186 \\ 1/6 & 1/3 & 1/6.1232 & 4 & 1/0.9743 & 1 & 0.3127 & 1/6 \\ 3 & 1/0.5038 & 1/6 & 1/0.1531 & 5 & 1/0.3127 & 1 & 0.3896 \\ 4 & 7 & 1/0.7461 & 8 & 1/0.1186 & 6 & 1/0.3896 & 1 \end{bmatrix}$$

(28)

Calculate the maximum eigenvalue and the corresponding right eigenvectors, we have $\lambda_{\text{max}}^{\text{LSM}} = 9.3177$, the eigenvectors are listed in Table 4.

To compare and analyze the results obtained by LSM and LAE methods, Table 4 summarizes five indicators, including the estimated values, the final weights of 8 factors, the final ranking, the corresponding maximum eigenvalues and the consistency ratios. It can be seen from Table 4 that the estimated value of unknown variable $x_4$ obtained by LAE is the same as the estimated value by LSM.

### Table 3

<table>
<thead>
<tr>
<th>Iter</th>
<th>$f(x)$</th>
<th>Iter</th>
<th>$f_1(x)$</th>
<th>Iter</th>
<th>$f_2(x)$</th>
<th>Variables $x_i$</th>
<th>Estimated values</th>
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one obtained by LSM. In addition, although most of the other estimated values obtained by both methods are slightly different, their integer approximated values within 9-point scale are equal to each other. Take the estimated values of unknown variables $x_2$, $x_3$, $x_7$ and $x_{10}$ obtained by LAE and LSM methods as examples, the corresponding values and integer approximated values are $1.8279/C_2$, $1.6536/C_2$, $0.4735/C_3$, $0.077/C_4$, $4.2863/C_5$, $6.1232/C_5$, $0.5264/C_6$, $0.7461/C_6$, $0.5320/C_7$, $0.5027/C_7$, $0.1378/C_8$, $0.1531/C_8$, $0.8928/C_9$, $0.9743/C_9$, $0.1111/C_{10}$, $0.1186/C_{10}$, $0.2902/C_{11}$, $0.3127/C_{11}$, $0.4232/C_{12}$, $0.3896/C_{12}$.

5. Conclusions

In this paper, the GMIBM is adapted and extended to estimate the missing comparisons and improve the consistency ratio at the same time. Specifically, the missing comparisons are first filled in by some unknown variables to obtain a revised 'complete' matrix, then construct geometric mean induced bias error matrix by the proposed model. Subsequently, two methods are provided to find the solution of unknown variables: (1) construct an overdetermined system of equations to solve the solution of variables; (2) construct an optimization problem either by the least absolute error (LAE) or by the least square method (LSM) to find the solutions. Different from the existing models for estimating the missing comparisons, our model only requires the original information of incomplete comparison matrix, and is independent of the weights. The correctnesses of two Corollaries of GMIBM are proved mathematically. One $4 \times 4$ incomplete matrix and an $8 \times 8$ high order incomplete matrix of emergency assessment are used to demonstrate the proposed models. The results show that the proposed models are not only capable of completing missing values, but also can efficiently improve the matrix consistency at the same time.
Although the proposed model can effectively estimate the missing comparisons in the simulated incomplete comparison emergency decision matrices, which may be one of the effective ways to make a fast emergency decision making by ignoring some unimportant pairwise comparisons or the comparisons beyond the decision maker’s capability, it remains to be validated by more real world emergency fast decision case studies that how fast the proposed model could improve the speed of response in real implication. In addition, it is noted that the limitation of the proposed model is that it can only be used in such case that the pairwise comparison technique is used to collect the experts’ judgments, especially when the AHP/ANP is used in the emergency management. Besides, the validity of the partial judgments provided by emergency experts could impact on the speed and effectiveness of an emergency response by the proposed method in practice.

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