High resolution numerical schemes for solving kinematic wave equation

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SUMMARY

This paper compares the stability, accuracy, and computational cost of several numerical methods for solving the kinematic wave equation. The numerical methods include the second-order MacCormack finite difference scheme, the MacCormack scheme with a dissipative interface, the second-order MUSCL finite volume scheme, and the fifth-order WENO finite volume scheme. These numerical schemes are tested against several synthetic cases and an overland flow experiment, which include shock wave, rarefaction wave, wave steepening, uniform/non-uniform rainfall generated overland flows, and flow over a channel of varying bed slope. The results show that the MacCormack scheme is not a Total Variation Diminishing (TVD) scheme because oscillatory solutions occurred at the presence of shock wave, rarefaction wave, and overland flow over rapidly varying bed slopes. The MacCormack scheme with a dissipative interface is free of oscillation but with considerable diffusions. The Godunov-type schemes are accurate and stable when dealing with discontinuous waves. Furthermore the Godunov-type schemes, like MUSCL and WENO scheme, are needed for simulating surface flow from spatially non-uniformly distributed rainfalls over irregular terrains using moderate computing resources on current personal computers.

KEYWORDS:
Kinematic wave equation
Godunov-type scheme
MacCormack scheme
Shock wave
Rarefaction wave
Rainfall-runoff overland flow

1. Introduction

The kinematic wave equation was first developed by Lighthill and Whitham (1955). The equation is based on the assumptions that the acceleration term and the pressure gradient term in the momentum equation are negligible, so that the energy slope is equal to the bottom slope. The kinematic wave model is commonly used to simulate the overland flow (Ponce, 1991; Singh, 2001). Henderson (1966) showed that natural flood waves behave nearly the same as the kinematic wave in steep slopes ($S_0 > 0.002$). Vieira (1983) concluded that the kinematic wave equation can be used on natural slopes with the kinematic wave number, $k \gg 50$. Ponce (1991) compared the kinematic wave equation with the unit hydrograph as a practical method of overland flow routing. Singh (2001) concluded that the kinematic wave equation is applicable to surface water routing, vadose zone hydrology, riverine and coastal processes, erosion and sediment transport, etc.

The kinematic wave equation is a first-order hyperbolic partial differential equation (PDE). For a hyperbolic equation, the disturbance will travel along the characteristics of the equation in a finite propagation speed. This feature distinguishes the hyperbolic equations from elliptic and parabolic equations. On the other hand, the kinematic wave equation also belongs to a kind of equations called conservation laws (LeVeque, 2002; Toro, 2009). Since the flux term is a nonlinear function of conservative variables, the solution does not propagate uniformly but deforms as it evolves. Even the initial conditions are continuous and smooth, the hyperbolic conservation laws can develop discontinuities in the solution, for example, shock waves.

Both shock and rarefaction waves are the intrinsic features of hyperbolic equations. Lighthill and Whitham (1955) discussed the formations of shock wave and rarefaction wave. Kibler and Woolhiser (1970) investigated the structure and general properties of shock waves and developed a numerical procedure for shock fitting. Eagleson (1970) found that using non-uniform flow depth as initial condition, non-uniform rainfall in the source term, or increasing inflows as the boundary condition may cause the formation of kinematic shock wave. Borah et al. (1980) presented the propagating shock-fitting scheme (PSF) to simulate overland flow with shock waves. Singh (2001) found three factors that affect the shock wave formation: (1) initial and boundary conditions; (2) lateral inflow and outflow; and (3) watershed geometric characteristics. Due to the complex geometry, non-uniform roughness
and non-uniform rainfall pattern, it is impossible to derive a general analytical solution for the kinematic wave equation. Singh (2001) summarized three numerical techniques for solving the kinematic wave equation: (1) method of characteristic, (2) Lax–Wendroff finite difference method, and (3) finite element method. Numerical diffusion and numerical dispersion were observed when using the finite difference schemes (Ponce, 1991). Kazeyerılmaz-Alhan et al. (2005) evaluated several finite difference schemes for solving kinematic wave equation: the linear explicit scheme, the four-point Pressimmann implicit scheme, and the MacCormack scheme. The study (Kazezyılmaz-Alhan et al., 2005) found the MacCormack scheme is better than the four-point implicit finite difference scheme for shock capture. However, Kazeyerılmaz-Alhan et al. (2005) did not explicitly examine the dispersion occurred at the shock and rarefaction waves from non-uniformly distributed rainfall. The stability of the classical MacCormack scheme at the presence of shock and rarefaction wave remains unknown.

Recently, the Godunov-type finite volume method has been widely used in solving shallow water equations (LeVeque, 2002; Toro, 2009) because of its wide applicability, strong stability, and high accuracy. One of the most popular Godunov-type methods is a second-order, TVD (Total Variation Diminishing) scheme, namely the MUSCL (Monotone Upstream-centered Schemes for Conservation Laws) scheme (van Leer, 1979). The MUSCL scheme is a high-resolution scheme because (1) the spatial accuracy of the scheme is equal to or higher than second order; (2) the scheme is free from numerical oscillations or wiggles; (3) high-resolution is produced around discontinuities. In general, the high-resolution schemes are considered as tradeoffs between computational cost and desired accuracy (Harten, 1983; Toro, 2009). Another popular but relatively new method is the high-order WENO (Weighted Essentially Non-Oscillatory) finite volume scheme (Shu, 1999). High-order means the order of accuracy is equal to or higher than the third-order. According to Shu (2009), the WENO scheme is suitable for the complicated problems, such as flow having both shocks and complicated smooth structures (e.g., small perturbation). Although the computational cost of high-order WENO scheme can be three to ten times than a second-order high-resolution scheme, it is still preferable because of its high-order accuracy in both time and space. The applications of those two high resolution schemes to solve the kinematic wave equation have not been studied. Whether or not these finite volume schemes have advantages over the commonly used finite difference schemes are examined in this paper.

This study compares the Godunov-type finite volume method using MUSCL scheme and WENO scheme with the finite difference method using MacCormack scheme. The paper is organized as follows: Section 2 introduces the kinematic wave equation and its analytical solutions; Section 3 discusses the numerical schemes: the MacCormack scheme, the MUSCL scheme and the WENO scheme; Section 4 shows the results of typical test cases. Finally, several concluding remarks are given in Section 5.

2. Governing equations

The one-dimensional kinematic wave equation for flows over a slope is given by (Eagleson, 1970; Lighthill and Whitham, 1955):

$$\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = i_0$$  \hspace{1cm} (1)

where $h$ is the depth of flow; $q$ is the discharge per unit width; $i_0 = i - f$ is the rain excess; $i$ is the intensity of rainfall; $f$ is the infiltration rate; $t$ is the time; $x$ is the downslope distance.

For the overland flow, the discharge $q$ is defined as:

$$q = 2h^m$$  \hspace{1cm} (2)

where $m$ is the exponential, and $\alpha$ is the coefficient. For fully turbulent flow, the coefficients are given by Ponce (1989):

$$\alpha = \frac{1}{n} \sqrt{S_0}, \quad m = \frac{5}{3}$$  \hspace{1cm} (3)

where $n$ is the Manning’s roughness coefficient; $S_0$ is the bottom slope. It is obvious that the flux function $q(h)$ is a convex function (Jacovkis and Tabak, 1996; Toro, 2009) because the second order derivative is positive:

$$\frac{d^2 q}{dh^2} = \alpha m(m - 1)h^{m-2} > 0, \quad \text{for } h > 0$$  \hspace{1cm} (4)

The analytical solution for Eq. (1) has been found by Eagleson (1970) in which the outflow hydrograph is a function of rainfall intensity and the time of concentration.
3. Numerical schemes

Since the kinematic wave equation is a nonlinear hyperbolic partial differential equation, different numerical schemes exhibit different amounts of numerical diffusion and dispersion depending on the nature of schemes. Numerical diffusion often presents itself as the attenuation of the kinematic wave, while numerical dispersion is responsible for oscillations or negative outflows near large surface gradients. To provide comparisons with classical finite difference schemes, a study of two high-resolution Godunov-type finite volume schemes, the MUSCL finite volume scheme and the WENO finite volume scheme, is presented in this paper. All the selected schemes are explicit, but differ in the order of accuracy. The MUSCL scheme is a second-order scheme while the WENO scheme is fifth-order.

3.1. MacCormack finite difference scheme

The MacCormack scheme (MacCormack, 2003) is a commonly used finite difference scheme to solve hyperbolic PDEs. This two-step scheme is second-order accurate in both time and space. Compared to the first-order scheme, the MacCormack scheme does not introduce numerical diffusion in the solution. However, the numerical dispersion can be introduced in the region of large surface gradients. The MacCormack scheme is also a variation of the two-step Lax–Wendroff scheme. It includes two steps: a predictor step followed by a corrector step. The predictor step uses forward difference approximations while the corrector step uses backward difference schemes.

The time step used in the predictor step is \( \Delta t \) in contrast to \( \Delta t/2 \) used in the corrector step. The discretized kinematic wave equation using the MacCormack scheme is below:

**Predictor step:**

\[
\frac{h_{i+1}^{n+1} - h_i^n}{\Delta t} = \frac{\partial h}{\partial x}(q_{i+1} - q_i) + i_0 \Delta t
\]

**Corrector step:**

\[
\frac{h_{i+1/2}^{n+1} - h_{i+1/2}^n}{\Delta t} = \frac{\Delta t}{\partial x} \left[ \frac{\partial h}{\partial x}(q_{i+1/2} - q_{i-1/2}) + i_0 \Delta t \right]
\]

where subscript \( i \) is the spatial index; superscript \( n \), \( n+1 \), and \( n+1 \) are the temporal indices; \( \Delta t \) is the time step; \( \Delta x \) is the space step.

The classical MacCormack scheme creates spurious oscillations at the fronts of shock and rarefaction waves (Garcia-Navarro et al., 1992; Macchione and Morelli, 2003). A remedy is to smooth out the oscillations using the dissipative interface (DI) after the corrector step (Abbott and Minnis, 1998). Mathematical formulation for the DI is as follows:

\[
\begin{align*}
\frac{h_{i+1}^{n+1}}{h_{i+1/2}^{n+1}} &= \frac{\partial h_{i+1/2}}{\partial x}(q_{i+1/2} - q_{i-1/2}), & \text{for internal nodes} \\
\frac{h_{i+1}^{n+1} + \partial h_{i+1}}{h_{i+1/2}^{n+1}} &= \frac{\partial h_{i+1/2}}{\partial x}(q_{i+1/2} - q_{i-1/2}), & \text{for left boundary node} \\
\frac{h_{i+1}^{n+1} + \partial h_{i+1}}{h_{i+1/2}^{n+1}} &= \frac{\partial h_{i+1/2}}{\partial x}(q_{i+1/2} - q_{i-1/2}), & \text{for right boundary node}
\end{align*}
\]

where \( h_{i+1/2} \) is the averaged solution; \( \theta \) is the coefficient for the dissipative interface.

3.2. MUSCL finite volume scheme

The MUSCL scheme was introduced by van Leer (1979). It is the first second-order TVD Godunov-type finite volume scheme. MUSCL uses piecewise linear approximation to reconstruct the depths at the interfaces of cells:

\[
h_{i+1/2} = h_i + \nabla h \cdot r_{i+1/2}
\]

where \( h_i \) is flow depth at the center of cell \( i \); \( h_{i+1/2} \) is the reconstructed depth at the interface \( i + 1/2 \); \( \nabla h_i \) is the limited depth gradient of cell \( i \); \( r_{i+1/2} \) is the distance from the cell center to the interface \( i + 1/2 \). The limited gradients can be calculated by several slope limiters (e.g., Minmod, Superbee, etc.). The van Leer limiter (Causon et al., 2000; van Leer, 1974) is used in the study:

\[
\nabla h_i = \frac{\nabla h_i^+ + \nabla h_i^-}{2}
\]

where \( \nabla h_i^+ = \frac{h_i+h_{i+1}}{\Delta x} \) and \( \nabla h_i^- = \frac{h_i+h_{i-1}}{\Delta x} \). The local Lax–Friedrichs (LF) method (LeVeque, 2002) is used to calculate the fluxes across the cell interfaces:

\[
Q_{i-1/2} = \frac{1}{2} [q_{i-1} + q_i - a_{i-1/2}(h_i - h_{i-1})]
\]

where \( c \) is the velocity and defined in Eq. (19).

3.3. WENO finite volume scheme

The first WENO scheme was provided by Liu et al. (1994), Jiang and Shu (1996) presented a general framework to construct arbitrary high order WENO schemes. Details about WENO schemes can be found in Shu (1999, 2009). Instead of using the piecewise linear reconstruction procedure in the MUSCL scheme, the WENO finite volume scheme uses the WENO reconstruction procedure to obtain an approximated function value at the cell interface. The WENO reconstruction procedure consists of four steps:

**Step 1.** Calculate the smoothness indicators:

\[
\begin{align*}
\beta_1 &= \frac{1}{\Delta x} (h_{i+1} - 2h_{i} + h_{i-1}) \\
\beta_2 &= \frac{1}{\Delta x} (h_{i} - h_{i-1}) \\
\beta_3 &= \frac{1}{\Delta x} (h_{i+1} - 2h_{i} + h_{i+2})
\end{align*}
\]

**Step 2.** Calculate the third-order approximations at cell interfaces:

\[
\begin{align*}
h_{i+1/2}^{[1]} &= \frac{1}{3} h_i + \frac{2}{3} h_{i+1} \\
h_{i+1/2}^{[2]} &= \frac{1}{3} h_i + \frac{2}{3} h_{i+1} + \frac{1}{3} h_{i+2}
\end{align*}
\]

**Step 3.** Calculate the nonlinear weights:

\[
w_j = \frac{\bar{w}_j}{w_1 + w_2 + w_3}, \quad \text{with} \quad \bar{w}_j = \frac{\gamma_j}{(\varepsilon + \beta_j)^{\gamma_j}}, \quad j = 1, 2, 3
\]
where \( \epsilon = 10^{-4} \) for actual calculations; \( \gamma_j \) is the linear weights, and is given by:

\[
\gamma_1 = \frac{1}{16}, \quad \gamma_2 = \frac{5}{8}, \quad \gamma_3 = \frac{5}{16}
\]  

Step 4. Calculate the fifth-order approximation as a convex combination of the three third-order approximations:

\[
h_{i+1/2} = w_1 h_{i+1/2}^{(1)} + w_2 h_{i+1/2}^{(2)} + w_3 h_{i+1/2}^{(3)}
\]

(17)

The derivative in time is discretized by the third-order TVD Runge–Kutta method (Gottlieb and Shu, 1998), which is a three-step method:

\[
\begin{align*}
h^{(1)} & = h^n + \Delta t \cdot L(h^n) \\
h^{(2)} & = \frac{3}{4} h^n + \frac{1}{4} h^{(1)} + \frac{3}{4} \Delta t \cdot L(h^{(1)}) \\
h^{n+1} & = \frac{1}{3} h^n + \frac{2}{3} h^{(2)} + \frac{2}{3} \Delta t \cdot L(h^{(2)}) 
\end{align*}
\]

(18)

where the operator \( L(h) \) is given by Eq. (12).

4. Results

This section evaluates the selected schemes in a variety of flow conditions because a given scheme does well for a test case does not guarantee it will do well for another test case. In order to evaluate the performances of these schemes, it is necessary to choose test cases with exact solutions. For this reason, the one-dimensional propagations of a shock wave and a rarefaction wave are selected. Then the behaviors of these schemes are tested by a wave steepening example. In addition, two synthetic rainfall-runoff cases and one overland flow over a varied slope are selected to demonstrate the applicability of those schemes. Finally, a real experimental case is simulated to test the performance of these schemes. The following abbreviations are used in the figures: MUSCL stands for the results by using the MUSCL scheme; WENO for the WENO scheme; MC for the MacCormack scheme; MC-DI for the MacCormack scheme with a dissipative interface. \( \theta = 0.25 \) is the coefficient for the dissipative interface used in all the test cases.

4.1. Shock wave

The shock wave and rarefaction wave are the intrinsic features of the hyperbolic equations, and thus the kinematic wave equation. The celerity of kinematic wave is defined as (Lighthill and Whitham, 1955):

\[
c = \sqrt{2m}h^{n-1} = mv
\]

(19)

where \( v \) is flow velocity. Consider the following initial-value problem for the kinematic wave equation:

\[
h(x, 0) = \begin{cases} 
  h_l, & \text{if } x < x_0 \\
  h_R, & \text{if } x > x_0 
\end{cases}
\]

(20)

If we assume \( h_l > h_R \), we will have \( c_l > c_R \). This means that a shock wave will arise. The exact shock wave solution for the kinematic wave equation is:

\[
h(x, t) = \begin{cases} 
  h_l, & \text{for } (x - x_0)/t < S \\
  h_R, & \text{for } (x - x_0)/t > S 
\end{cases}
\]

(21)

where \( S \) is the shock wave speed. For the kinematic wave equation, according to the Rankine-Hugoniot jump condition (Toro, 2009), \( S \) is equal to:

\[
S = \frac{\sqrt{c_l} h_l - \sqrt{c_R} h_R}{h_R - h_l}
\]

(22)

In the study, the following parameters are used: the channel length \( L = 10.0 \) m; the bed slope \( S_0 = 0.0016 \); the Manning’s coefficient \( n = 0.025 \) s/m^{1/3}; the simulation time \( t = 3.5 \) s; the time step \( \Delta t = 0.01 \) s; the grid spacing \( \Delta x = 0.1 \) m. The initial condition is: \( h_l = 1.0 \) m and \( h_R = 0.5 \) m. The boundary conditions are zero-depth gradient at both the entrance and the exit of the channel.

The flow depth profiles calculated by the numerical schemes and the exact solution are plotted in Fig. 1. The results show that the numerical diffusion for flow depth near the front of shock wave is small for the MUSCL and WENO schemes. But the results of flow depth from the MacCormack scheme are oscillatory. The wiggled surface oscillations are largest at the discontinuous wave front, and the solutions eventually diverge. No oscillations are present using the MacCormack scheme with the dissipative interface. But, the results are not accurate because the dissipative interface has smoothed out the sharp front of the shock wave.

To determine if a numerical scheme will introduce oscillations in the solution, the total variation defined in Eq. (23) can be used.

\[
TV(h) = \sum_{i=-\infty}^{\infty} |h_i - h_{i-1}|
\]

(23)

where \( TV(h) \) is the total variation of flow depth, \( i \) is the index of computational cells. If the total variation is increasing with time, the numerical scheme will introduce oscillations; otherwise, it preserves the monotonicity of initial monotone functions (LeVeque, 2002). To avoid oscillatory solutions, the total variation needs to decrease or remain constant with time. The total variations of flow depth from those schemes are shown in Fig. 2, which indicate that the total variation increases with time using the MacCormack scheme, while it remains constant for MC-DI, MUSCL and WENO schemes. This implies that the MacCormack scheme will introduce oscillations at the presence of shock waves, but MC-DI, MUSCL and WENO schemes can preserve the functional properties of the shock wave. Therefore, the solutions using the MacCormack scheme are
oscillatory, and those using the MC-DI scheme are inaccurate for capturing the shock wave.

4.2. Rarefaction wave

Reconsider the initial-value problem described in Eq. (20), and assume \( h_L < h_R \), we have \( c_L < c_R \). This time, instead of generating a shock wave, a rarefaction wave is generated near the discontinuity since the celerity at the head of the discontinuity is greater than that at the tail and, consequently, the discontinuity continually expands as it propagates. For the kinematic wave equation, the exact solution of rarefaction wave is:

\[
\begin{align*}
    h &= \begin{cases} 
        h_L, & \text{for } \frac{\Delta h}{\Delta t} < c_L \\
        h_L + \left( c_L - h_L \right) \frac{h_L - h_R}{c_L - c_R}, & \text{for } c_L \leq \frac{\Delta h}{\Delta t} \leq c_R \\
        h_R, & \text{for } \frac{\Delta h}{\Delta t} > c_R
    \end{cases}
\end{align*}
\]

(24)

The parameters used in the rarefaction test case are summarized as: the channel length \( L = 10.0 \text{ m} \); the bed slope \( S_0 = 0.0016 \); the Manning’s coefficient \( n = 0.025 \text{ s/m}^{1/3} \); the simulation time \( t = 3.0 \text{ s} \); the time step \( \Delta t = 0.01 \text{ s} \); and the grid spacing \( \Delta x = 0.1 \text{ m} \). The initial condition is: \( h_L = 0.5 \text{ m} \) and \( h_R = 1.0 \text{ m} \). The boundary conditions are zero-depth gradient at both ends of the channel.

The calculated flow depth profiles are plotted in Fig. 3. The results are similar to the shock wave case: the MacCormack scheme generated notable oscillations at the tail of the rarefaction wave. Taylor et al. (1972) tested the MacCormack scheme for solving the Burgers’ equation and found that the MacCormack scheme is unstable for rarefaction wave under some conditions. This study finds that the same phenomenon occurred to the kinematic wave equation. Since the oscillations occur at the tail of the rarefaction wave, where flow depth is smaller, unrealistic negative flow depths can be induced at the tails of rarefaction waves. This can also be proved by calculating the total variation of flow depth as shown in Fig. 4. The total variations from MC-DI, MUSCL and WENO schemes are constant. But the values from the MacCormack scheme increase suddenly to a peak value and then gradually decrease to a constant value (about 0.9) greater than the initial total variation (=0.5). Although the total variation shows a decreasing trend, the scheme cannot keep the total variation not to increase with time for the entire simulation time. Therefore, oscillations occur at the discontinuous front, the tail of rarefaction wave. Therefore, for the rarefaction wave, the MacCormack scheme is not as stable as the MC-DI, MUSCL or WENO scheme. Both the shock and rarefaction wave test cases suggest that the MacCormack scheme is not a Total Variation Diminishing (TVD) scheme, and the oscillations will be generated near the discontinuous wave fronts. Although the MC-DI scheme has eliminated spurious oscillations in the solution, it also smears out the discontinuous wave fronts by introducing redundant numerical diffusion to the solution. Therefore, the MUSCL scheme and WENO schemes should be used to simulate unsteady flows with shock and rarefaction waves.

4.3. Wave steepening

One of the prominent features of the hyperbolic equation is that discontinuities will be generated even the initial water surface is smooth. So it is important for a scheme to preserve the stability...
and sharpness of discontinuous fronts in a simulation. This test case is to test this behavior of these schemes, especially the ability of anti-diffusion.

Here are the parameters in the test case: the channel length \( L = 10.0 \, \text{m} \); the bed slope \( S_0 = 0.0016 \); the Manning’s coefficient \( n = 0.025 \, \text{s/m}^{1/3} \); the simulation time \( t = 3.5 \, \text{s} \); the time step \( \Delta t = 0.01 \, \text{s} \); and the grid spacing \( \Delta x = 0.1 \, \text{m} \). The initial condition is:

\[
h = \max\left[0.5, 1.0 - (x - 1.0)^2\right]
\]

This initial condition will create a parabolic perturbation in the channel as shown in Fig. 5. The boundary conditions are periodic flow depths at both ends of the channel.

The parabolic perturbation will produce a “wave steepening” effect in the domain (Henderson, 1966; Toro, 2009). The reason is that the celerity of the perturbation is faster than the ambient fluid. The head of the perturbation is a compressive region while the tail of the perturbation is an expansive region. This perturbation is a combination of a shock wave (head) and a rarefaction wave (tail). This scenario is detrimental to the MacCormack scheme because the oscillations occurred behind the shock wave will drown out the parabolic perturbation immediately (Fig. 6). Consequently, the MacCormack scheme cannot converge to a stable solution, but amplified oscillations. This further proves that the MacCormack scheme is an unstable scheme for flows of discontinuous waves. The results of MC-DI, MUSCL and WENO schemes are plotted in Fig. 7. After 3.5 s, the fronts of the perturbation remain sharp using the MUSCL and WENO schemes, whereas the results from the MC-DI scheme cannot preserve the sharp front because of the numerical diffusion.

### 4.4. Uniform rainfall-runoff overland flow

The uniform rainfall-runoff overland flow can be solved by the method of characteristics (Eagleson, 1970; Kazezyılmaz-Alhan et al., 2005). At a given rainfall excess in a specified duration, the outflow hydrograph \( q = q_a, x = L \) can be solved analytically. Assuming a constant rainfall excess \( i_0 \) and an initial zero flow depth,

\[
h = 0(t = 0 \text{ and } 0 \leq x \leq L)
\]

and the boundary condition at the channel entrance,

\[
h = 0(t > 0 \text{ and } x = 0)
\]

The solutions of two possible outflow hydrographs are summarized below:

**Case 1 \( t_r \geq t_c \):**

\[
q_r = zh_r^m, \text{ to solve } h_r \text{ use: } \begin{cases} h_l = i_0 t_r, & \text{ for } t \leq t_r \\ h_l = i_0 t_c, & \text{ for } t_r < t \leq t_r \\ L = zh_r^{m^{-1}}[ht_0^{-1} + m(t - t_r)], & \text{ for } t > t_r \\ \end{cases}
\]

**Case 2 \( t_r < t_c \):**

\[
q_r = zh_r^m, \text{ to solve } h_r \text{ use: } \begin{cases} h_l = i_0 t_r, & \text{ for } t \leq t_r \\ h_l = i_0 t_p, & \text{ for } t_r < t \leq t_p \\ L = zh_r^{m^{-1}}[ht_0^{-1} + m(t - t_r)], & \text{ for } t > t_p \\ \end{cases}
\]

where \( t_r \) is the duration of rainfall, and \( t_c \) is the time of concentration:

\[
t_c = \left(\frac{L t_0^{1-m}}{2}\right)^{1/m}
\]

and \( t_p \) is defined as:

\[
t_p = t_r + \frac{t_c - t_r}{m}, \quad t_r = \frac{L}{2zh_0^{-m}}, \quad h_0 = i_0 t_r
\]

In Kazezyılmaz-Alhan et al. (2005)’s hypothetical experiment, the duration of rainfall is longer than the time of concentration \( t_r \geq t_c \). The experimental parameters are summarized here: the channel length \( L = 182.88 \, \text{m} \); the bed slope \( S_0 = 0.0016 \); the Manning’s coefficient \( n = 0.025 \, \text{s/m}^{1/3} \); the duration of rainfall \( t_r = 0.5 \, \text{h} \); the rainfall excess \( i_0 = 50.8 \, \text{mm/h} \); simulation time \( t = 1 \, \text{h} \); the time step \( \Delta t = 1.0 \, \text{s} \); and the grid spacing \( \Delta x = 1.83 \, \text{m} \). The initial condition is \( h = 0 \, \text{m} \). The boundary conditions are zero-depth gradient at the outlet and zero depth at the inlet.
Fig. 8 plots the outflow hydrographs calculated by the numerical schemes and the exact solution calculated by Eq. (28). Since there is no discontinuity/perturbation in the domain, the results obtained by the MacCormack, MUSCL and WENO schemes are very close to the exact solution. The results using the MC-DI scheme showed remarkable numerical diffusions than those using other schemes. The comparisons of the peak of the hydrographs show that the WENO scheme yields the closest results to the exact solution without any oscillation. When the outflow discharge reaches the peak, flow in the channel reaches steady state. To check if the mass is conserved in the simulation, the differences between the total outflow volume and the total rainfall volume are calculated, and then non-dimensionalized by the total rainfall volume. The percentage flow differences relative to the total rainfall are 0.65% for the MacCormack scheme, 0.59% for the MC-DI scheme, 0.35% for the MUSCL scheme, and 0.30% for the WENO scheme, respectively. All schemes preserve mass conservation very well at the absence of discontinuous waves. The MUSCL scheme and WENO scheme are slightly more conservative than the MacCormack scheme.

4.5. Steady non-uniform rainfall-runoff overland flow

To compare the stability of those schemes, a steady non-uniform rainfall-runoff case, Case I in Moramarco and Singh (2002), was simulated. The simulated plane is 100 m long with a slope of \( 1.5 \times 10^{-3} \), and the Manning’s roughness coefficient is 0.013 s/m\(^{1/3}\). The following steady non-uniform rainfall excess is used in this case (Fig. 9):

\[
i_0 (10^{-5} \text{ m/s}) = \begin{cases} 
4.43, & x \leq 50 \text{ m} \\
0.0, & x > 50 \text{ m}
\end{cases}
\]  

(32)

The simulated results are showed in Figs. 10 and 11. The depth of runoff generated by the non-uniform rainfall is not smooth and a shock wave is generated near the front of runoff. Oscillations occurred in the solution of the MacCormack scheme. Solutions by the MC-DI, MUSCL and WENO schemes are stable, and free of oscillations. But, only the results using the MUSCL and WENO schemes can capture the sharp front of shock wave generated by the runoff. The computation is run with a DELL XPS notebook (Intel i5 M450 2.4 GHz CPU with 4 GB memory). The programs are developed using Matlab 2011b running on Microsoft Windows 7.0 OS. For this simulation, the CPU times using the MacCormack, MC-DI, MUSCL, and WENO schemes are 13.4 s, 13.7 s, 14.1 s, and 27.3 s,
respectively. As expected, the MacCormack scheme runs the fastest, while the WENO scheme is two times slower than the MUSCL scheme.

4.6. Rainfall-runoff over non-uniform overland slope

Since overland flow always travels in channels of varying slopes, this requires the numerical scheme for solving the kinematic wave equation to be valid for changing bed slopes. Prior test cases have shown that the MacCormack scheme is unstable for flows with discontinuous waves (e.g., shock, rarefaction). Therefore, the MacCormack scheme is not applicable to non-uniform rainfall on a uniform slope. To verify its applicability on a changing slope, this study hypothesizes an experiment of rainfall and overland flow in a watershed having the same experimental conditions as in Kazezyılmaz-Alhan et al. (2005) except the upstream half of the channel has a slope different from the downstream half (\( S_0 = 0.0016 \)). This study adopted \( S_0 = 0.016 \) as the bed slope for the upstream half of the channel. The simulated water surfaces are shown in Fig. 12. As the slope in the upstream half channel increases to 0.016, ten times of the downstream slope, the MacCormack scheme generated severe oscillations. This test verifies that the MacCormack scheme is not only unstable for shock and rarefaction waves but also for flows on rapidly varying slopes.

4.7. Overland flow experiment (Schreiber, 1970)

This laboratory experiment was conducted in the Erosion Laboratory, Palouse Conservation Field Station, USDA-ARS-SWC (Schreiber, 1970). The experiment was conducted on a sloped plane to mimic one dimensional overland flow. During the experiment, the rainfall is generated by an array of nozzles. The runoff from the plane was drained into a tank and the discharge was measured by a strain gauge. The length of the plane is 4.88 m. The plane slope is 0.0465, and the Manning’s roughness coefficient is 0.0125 s/m\(^{1/3}\). The intensity of rainfall is 27 mm/h with duration of 4.0 min. For the model, the simulation time is 8.0 min. The upstream boundary condition is zero flux, the same as the solid wall boundary condition. The downstream boundary condition is defined by the characteristic equation (Eq. 19).

The simulated hydrographs by using the above mentioned schemes are compared with the measured ones in Fig. 13. Fig. 13(a) shows the simulated results are nearly the same using different schemes. A detailed plot in Fig. 13(b) shows the MacCormack scheme over-predicted the discharge. However, none of the simulated results matched the experimental observations. This attributes to the assumption of kinematic wave equation, which has a smaller celerity than the dynamic wave, and unable to capture the shock wave front. Therefore, the proposed MUSCL and WENO solver for the kinematic wave equation is valid only for overland flow when the kinematic wave assumption is valid.

To evaluate the accuracy of those numerical schemes, the root-mean-square error (RMSE) of three test cases, i.e. the shock wave case, the rarefaction wave case, and the rainfall-runoff case, are calculated by comparing to the exact solutions in Table 1. The RMSE showed that the MacCormack scheme with the dissipative interface has the maximum error, although its results are free of oscillation. This implies that the dissipative interface brought in considerable numerical diffusions to the solutions. On the other hand, the MUSCL and the WENO scheme are not only free of oscillation, but also yielded more accurate results than the MacCormack scheme.

In summary, the testing cases listed in Sections 4.1-4.7 demonstrated the limitations of the MacCormack scheme for solving the
kinematic wave equation and proved the validity of high resolution schemes, the MUSCL and WENO schemes. Although the MacCormack scheme has been widely used to solve the kinematic wave equation, it should be avoided at the presence of shock/rarefaction waves, or flow over rapidly varying sloped land surfaces. The dissipative interface for the MacCormack scheme resulted in excessive oscillations in the solutions using the MUSCL scheme or the WENO scheme. The high-order WENO scheme shows the best resolution in all test cases. Although the computational costs are higher than that of the MacCormack scheme, to ensure numerical stability, the MUSCL and WENO schemes are the stable and accurate techniques for solving the kinematic wave equation.

5. Conclusions

In this study, the applicability of four numerical schemes (MacCormack scheme, MacCormack scheme with the dissipative interface, MUSCL scheme and WENO scheme) to solve the kinematic wave equation is investigated. The schemes are examined by using several test cases: shock wave, rarefaction wave, wave steepening, overland flow from uniform or non-uniform rainfall, overland flow over varying bed slopes. The results show that oscillations appeared in the solutions using the MacCormack scheme when the shock and rarefaction waves are present, or bed slopes changing rapidly. Since the MacCormack scheme is not a TVD scheme, the oscillations seriously affect the stability of the solution, and make the solution diverge. The dissipative interface can eliminate spurious oscillations seen in the MacCormack scheme, but it also brings in redundant numerical diffusions to solution. On the other hand, the MUSCL and WENO schemes consistently perform better than the MacCormack scheme. There is no oscillation in the solutions using the MUSCL scheme or the WENO scheme. The high-order WENO scheme shows the best resolution power in all test cases. Although the computational costs are higher than that of the MacCormack scheme, to ensure numerical stability, the MUSCL and WENO schemes are the stable and accurate techniques for solving the kinematic wave equation.

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