A new iterative method for solving multiobjective linear programming problem

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Abstract

In the paper we present a new iterative method for solving multiobjective linear programming problems with an arbitrary number of decision makers. The method is based on the principles of game theory. Each step of the method yields a unique solution which respects the aspirations of decision makers within the frame of given possibilities. Each decision maker is assigned an objective indicator which shows the reality of his aspiration and which may be used to define the strategy for the next step. The method can be easily extended to general (nonlinear) multiobjective programming problems but the numerical application would require further research on computational methods.

1. Introduction and motivation

The motivation for this work arises from a practical, frequently encountered problem: Several decision makers (we shall call them players) optimize their utilities at the same time and on the same constraint set (budget). They can achieve their aspirations at different optimal points. However, only one such point is available and the players are aware of that, which is known as the multiple objective programming problem (MOPP). Solving such a MOPP requires some kind of cooperation among the players. The goal can also be achieved by following the rules of a regulatory subject, if such exists.

The research literature on the cooperative game theory is extensive and our review will include only a select number of papers dealing with the problem of achieving cooperation when choosing the preferred solution in MOPPs. The impact of individual aspirations on promoting the cooperative level has been studied in [9,10,18]. The application of cooperative game theory in social sciences has been presented in [3,11,20,21]. Some specific problems arising from the application of cooperative game theory in physics have been studied in [4,16,17,19].

If the objective functions are linear and the budget is defined by the linear constraints, we are dealing with a multiobjective linear programming problem (MOLPP). There were many attempts at solving MOLPP and a variety of methods have been proposed. All of them require that the players (decision makers) participate in the selection of the preferred solutions. The first wide-ranging review of methods for solving MOLPP appeared in [2], following which many authors developed new methods to remove the shortcomings of the earlier ones. Different methods require different levels of players’ participation in the problem solving process. The existing methods are burdened by the complicated solving procedure when a larger number of players are involved.

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Over the last ten years such methods have also incorporated the game theory idea of equilibrium. However, all those methods can only be applied to solving real problems with considerable difficulty. Below is a brief overview of the methods which incorporate cooperative or/and noncooperative game theory ideas.

In [15], a hybrid multiobjective algorithm derived from game theory is applied to an integrated process planning and scheduling (IPPS) problem. Namely, the Nash equilibrium has been used in a game theory based algorithm to deal with multiple objectives. The algorithm is complex and incomprehensible to the players. This is the main disadvantage of the algorithm.

The authors in [1] developed a multiobjective game theory model (MOGM) for balancing economic and environmental concerns in reservoir watershed management and for assistance in decision making. In this case game theory is used as an alternative tool for analyzing the strategic interaction between economic development (land use and development) and environmental protection (water-quality protection and eutrophication control). The methodology is simple enough, but it is only applicable to problems involving two players.

In [5] a production model is considered as a multiobjective linear programming problem with multiple players. It is shown that a multi-commodity game arises from the multi-objective linear production programming problem with multiple decision makers. The characteristic sets in the game were obtained by finding the set of all the Pareto extreme points of the multiobjective programming problem. It is proven that the core of the game is not empty, and points in the core are computed using the duality theory of multiobjective linear programming problems. The least core and the nucleolus of the game are examined as well. The proposed methodology, however, is quite complicated and the players, who should be able to understand it and trust the results, can hardly understand do so.

The authors in [8] develop a multiobjective mixed integer linear programming model, devised to optimize the planning of supply chains using game theory optimization for decision making in cooperative and/or competitive scenarios. The multiobjective problem is solved using the Pareto frontier solutions, and both cooperative and noncooperative scenarios between supply chains were considered. This algorithm is designed for solving a specific problem and it is not generally applicable.

A method for multiobjective categorization based on the game theory and Markov process is proposed in [14]. The authors adopt Shapley value in coalitional games to measure the player’s satisfaction degree in a group. Next they present the concept of priority groups and an algorithm to combine small-size priority groups with large-size ones, which may improve the efficiency of calculating the player’s satisfaction degree. The complexity of this algorithm is its main disadvantage.

A good application of cooperative and noncooperative game theory has been presented in [12]. A difference between noncooperative and cooperative games is that cooperative game theory admits of binding agreements to choose a joint strategy in the mutual interest of those who agree.

In [13] the authors propose a realistic representation of a decision maker’s behavior by synthesizing games-against-nature and goal programming into a single framework. The proposed model has been illustrated by an example from the literature on mathematical programming models for agricultural-decision-making.

The idea of cooperative games is obviously very interesting and can be used for optimization of complex systems with multiple players. In fact, optimization of a complex system requires that the players take into account not only their individual goals but also the need for an optimal functioning of the whole system. The goal of this paper is to use game theory ideas to develop a simple method for solving MOLPP with any number of players.

Generally speaking, the problem to be solved is the following one: optimize of a multi-component system in which each component has its separate goal and the objectives are mutually conflicting. Each component of the system represents one player. In addition to the individual objectives of each system component there is a common goal that cannot be precisely measured: the optimal functioning of the whole system. The system is operating optimally if all components of the system are met at a satisfactory level with regard to the players. The satisfactory level is reached through the process of problem solving, where we use the strengths of game theory, and which allows us to achieve equilibrium solutions. The solving process stops when a desired (Nash) equilibrium is obtained. According to Martin J. Osborne: a Nash equilibrium is an action profile \( a^* \) with the property that no player \( i \) can do better by choosing an action different from \( a_i^* \), given that every other player \( j \) adheres to \( a_j^* \). (see [7], page 22).

Our aim is to build a simple method for determining optimal solutions MOLPP with multiple players, which uses the good ideas of cooperative games, and where the system operates optimally when all its components operate at a satisfactory level. The players are prepared to cut their aspirations in terms of achieving the goals of system components that they represent, if such behavior will contribute to improving the functioning of the whole system. The solving procedure should lead to the equilibrium solutions (Nash equilibrium).

2. Statement of the problem

In this paper we present a new method which efficiently solves MOLPP. The following problem will be considered: Let \( z_i(x) \), \( x \in \mathbb{R}^n \), be the given linear objective function for player \( i \) (\( P_i \)), i.e.

\[
z_i(x) = z_i(x_1, x_2, \ldots, x_n) = \sum_{j=1}^{n} c_{ij} x_j, \quad i = 1, 2, \ldots, k.
\]
where \( c_0 \) are real constants and \( k \) is the number of players. Let \( A = (a_{ij}) \in \mathcal{R}^{m \times n} \) and \( b \in \mathcal{R}^m \) be given matrix and vector, respectively. Here \( \mathcal{R}^{m \times n} \) denotes the spaces of real matrices of order \( m \times n \). We consider the following MOLPP

\[
\max_{x \in S} (z_1(x), z_2(x), \ldots, z_k(x)), \quad \text{where } S = \{x \in \mathcal{R}^n : x \geq 0, Ax \leq b\}.
\] (1)

Here the budget \( S \) is defined as an intersection of \( m \) linear constraints where \( \leq \) means that the constraints may be given by inequalities (\( \leq \) or \( \geq \)) or equalities (\( = \)). Obviously, \( S \) is a convex set in the first orthant of the vector space \( \mathcal{R}^n \). Usually, in practical applications, \( S \) is bounded and it contains the origin. If we introduce matrix \( C = (c_{ij}) \in \mathcal{R}^{k \times n} \), then MOLPP (1) can be stated by the following simple formulation

\[
\max_{x \in S} Cx.
\]

We assign the following \( k \) linear programming problems (LPP) to (1),

\[
\max_{x \in S} z_i(x), \quad i = 1, 2, \ldots, k.
\] (2)

We know that each of these LPP, for the bounded budget \( S \), has the unique solution \( z_i \) at (not necessary unique) optimal point \( x^0 \). We intend to find some kind of equilibrium where the players \( P_i, i = 1, 2, \ldots, k \) achieve their optimums at the same point \( \hat{x} \). For this purpose we proceed in the following way.

### 3. The new method

#### 3.1. One step of the method

Let \( d_i, i = 1, 2, \ldots, k \) be the desired aspiration level for \( P_i \), which means that \( P_i \) wants to achieve \( z_i(x) \geq d_i \). It is natural to assume \( d_i \leq \hat{z}_i \) because, on the budget \( S \), \( P_i \) cannot reach any level greater than \( \hat{z}_i \). But this assumption is not necessary (\( d_i \) can be any arbitrary number). Now, for given \( d_i, i = 1, 2, \ldots, k \), we define the set of desired aspiration levels (desired budget),

\[
D = \{x \in \mathcal{R}^n : x \geq 0, z_i(x) \geq d_i, \quad i = 1, 2, \ldots, k\}.
\] (3)

Note that at any \( x \in D \) all players realize their aspirations. Let \( \Gamma = S \cap D \). Three possible cases may occur:

1. \( \Gamma \) is an empty set. It means that at least one aspiration level is unreal (it is set too high) and it cannot be realized on the budget \( S \).
2. \( \Gamma \) is the single point \( \hat{x} \). In this case we have the solution of (1) at that point. At least one aspiration level is exactly achieved (there exists \( j \) such that \( z_j(\hat{x}) = d_j \)) and the others are achieved more than expected (\( z_i(\hat{x}) > d_i \)).
3. \( \Gamma \) is a non-trivial set (it contains at least two points). It means that (1) has the solution at any point in \( \Gamma \). The players may choose among them to find the best possible solution (e.g. by performing a certain optimization procedure).

Thus, we have the solution if \( \Gamma \) is not empty. It remains to solve (1) for \( \Gamma = \emptyset \) which will usually occur. In this case the players cannot realize their aspirations to the full desired extent. To solve the problem we will project the desired budget set towards the budget set until they touch each other. For this purpose we define the shifted desired budget,

\[
D_{\lambda} = \{x \in \mathcal{R}^n : x \geq 0, z_i(x) \geq \lambda d_i, \quad i = 1, 2, \ldots, k, \quad \lambda \geq 0\}.
\] (4)

This is the budget with shifted desired aspiration levels (\( \lambda d_i \) instead of \( d_i \)). Geometrically, \( D_{\lambda} \) is the homothetic transformation of \( D \) with the ratio \( \lambda \), which is also the ratio of similarity. Note that \( D_1 = D \) and \( D_0 \) is a cone with vertex at the origin. Now, if \( S \cap D = \emptyset \), we shall look for \( \lambda \) such that \( S \cap D_{\lambda} \neq \emptyset \). Two natural questions appear. Does such \( \lambda \) exist? If it does, how can we find it? We answer these questions below.

**Theorem 1.** Let \( S, D \) and \( D_{\lambda} \) be defined by (1), (3), and (4), respectively, where \( S \) is a bounded set and \( d_i \geq 0, i = 1, 2, \ldots, k \). Let \( \Gamma_{x} = S \cap D_{\lambda} \). If \( D \neq \emptyset \) and \( \Gamma_{x} \neq \emptyset \) then there exists \( \tilde{\lambda} \) such that

\[
\begin{cases}
\Gamma_{x} \neq \emptyset & \text{for } 0 \leq \lambda \leq \tilde{\lambda}, \\
\Gamma_{x} = \emptyset & \text{for } \lambda > \tilde{\lambda}.
\end{cases}
\]

Theorem 1 can be proved in different ways. We give an elementary topological proof (see also the geometric illustration in Fig. 1). Let \( \gamma \) be the smallest distance from set \( D \) (i.e. its border) to the origin. Then, \( \gamma \) is the smallest distance from set \( D \) to the origin. Since \( \gamma \) can be arbitrarily large and \( S \) is a bounded set, we conclude that there exists (enough large) \( \lambda = \mu \) such that \( \Gamma_{\mu} = \emptyset \). Thus, we have \( \Gamma_{0} \neq \emptyset, \Gamma_{\mu} = \emptyset \). The existence of \( \lambda \) follows from the fact that the border of \( D \) (and the corresponding part of \( \Gamma_{x} \)) continuously depends on \( \lambda \), and \( \Gamma_{x} \) is a closed, bounded set. The uniqueness of \( \lambda \) follows from the following implications:

\[
l_{\lambda} < \tilde{\lambda} \Rightarrow (z_1(x) \geq \lambda d_1) \Rightarrow z_1(x) \geq \lambda_1 d_1 \Rightarrow D_{\lambda_1} \subseteq D_{\lambda} \Rightarrow \Gamma_{\lambda_1} \subseteq \Gamma_{\lambda},
\]

that is, if \( \Gamma_{\lambda_1} = \emptyset \) then \( \Gamma_{\lambda} = \emptyset \) for \( \lambda \geq \lambda_1 \) and also if \( \Gamma_{\lambda_2} \neq \emptyset \) then \( \Gamma_{\lambda} \neq \emptyset \) for \( \lambda \leq \lambda_2 \).

What follows is another, constructive proof of Theorem 1: the method for computing \( \lambda \). For \( x \in \mathcal{R}^n, \lambda \in \mathcal{R} \) we define the following LPP which is assigned to (1),
\[
\max_{\lambda \in \mathbb{R}^n} \lambda, \quad \text{where} \quad G = \left\{ (x, \lambda) \in \mathbb{R}^{n+1} : x \geq 0, \lambda \geq 0, Ax \leq b, z_i(x) \geq \lambda d_i, i = 1, 2, \ldots, k \right\}.
\]

Since \( G \in \mathbb{R}^{n+1} \) is a closed, convex and bounded set, problem (5) has the unique solution \( \lambda \). Note that for a fixed \( \lambda \), \( G \) is reduced to \( \Gamma_\lambda \), which yields the assertion of Theorem 1.

Let us briefly analyze the problem (5) and its solution. Let \( (\bar{x}, \bar{\lambda}) \in \Gamma_\lambda \) be the optimal point. Note that \( \bar{x} \in S \) is not necessarily unique. Since \( (x, \lambda) \) belongs to the border of \( G \), for some \( i \in \{1, 2, \ldots, k\} \) we have \( z_i(x) = \lambda d_i \) (active constraints) and for the others we have \( z_i(x) > \lambda d_i \) (passive constraints). We define the indicators \( \lambda_i \),

\[
\lambda_i = \frac{z_i(\bar{x})}{d_i}, \quad i = 1, 2, \ldots, k,
\]

which show to what extent the aspiration level \( d_i \) of player \( P_i \) can be realized. Let us explain this a little better by the following example.

**Example 1.** Let \( k = 4 \) and \( \lambda_1 = 0.75, \lambda_2 = 0.6, \lambda_3 = 1.11, \lambda_4 = 0.9 \).

It means that \( P_1 \) can realize 75% of his aspiration level, \( P_2 \) 60%, \( P_3 \) 111% and \( P_4 \) 90%. Thus, \( \bar{\lambda} = \lambda_2 = 0.6 \), which means that all players can realize their aspiration levels to at least 60%. If we consider the constraints \( z_i(\bar{x}) \geq \lambda d_i, i = 1, 2, 3, 4 \), we see that for \( i = 2 \) the constraint is active \( (z_2(\bar{x}) = \lambda d_2 = 0.6d_2) \), while the others are the passive ones \( (z_i(\bar{x}) > \lambda d_i = 0.6d_i, i = 1, 3, 4) \).

Why cannot the optimal solution \( \bar{\lambda} = 0.6 \) be larger? Because \( d_2 \) is set too high. If \( P_2 \) decreases his level then \( \lambda \) will be larger. Therefore, the smallest indicators (which are equal to the optimal one and yield the active constraints) point to those players who set their levels too high compared to others and thus disable the other players to realize a higher level. We also see that the players \( P_1, P_3 \) and \( P_4 \) may increase their aspiration levels (until their constraints become active) and it will not affect the optimal values \( \bar{x} \) and \( \bar{\lambda} \). Generally, the alternative possible choice \( d_i' \) must satisfy is

\[
d_i' \leq \frac{\lambda_i}{\bar{\lambda}} d_i, \quad i = 1, 2, \ldots, k.
\]

In our example

\[
d_1' \leq \frac{0.75}{0.6} d_1 = 1.25d_1, \quad d_3' \leq \frac{1.11}{0.6} d_3 = 1.85d_3, \quad d_4' \leq \frac{0.9}{0.6} d_4 = 1.5d_4,
\]

We see that \( P_1 \) can increase his aspiration level by up to 25% which will not affect the realizations of other players. The same is true for \( P_3 \) and \( P_4 \), who can increase it by up to 85% and 50%, respectively. Thus, the relative distance between \( \lambda_i \) and \( \bar{\lambda} \),

\[
\frac{\lambda_i - \bar{\lambda}}{\bar{\lambda}} = \frac{\lambda_i}{\bar{\lambda}} - 1,
\]

shows to what maximum extent \( P_i \) could increase his aspiration level without endangering the others. Consequently, if the player has more modest (more realistic) requirements, the assigned indicator is larger and vice versa. Such players will realize their aspirations to the greater extent than the others. A player can realize even more than he aspires, in which case his indicator is greater than one (\( P_3 \) in our example, where \( \lambda_3 = 1.11 \)). The ancient wisdom is confirmed: the less you look for, the more you get.
3.2. Strategy

The problem (5) represents a step of the iterative method intended for solving the initial problem (1). If the players are satisfied with the obtained solution, then “the game is over”. If not, then they need to define a strategy for the next step, in which the solution will be improved. Here strategy means how the initial data (levels) for the next step are defined. The indicators $\lambda_i$ help players to choose the strategy which will ensure the desired improvements. The initial data may be redefined in two possible ways. The player $P_i$ can define:

1. New aspiration level $d_i$. The constraint $z_i(x) \geq \lambda d_i$ will participate in the definition of $D_i$ for the next step.
2. Absolute level $g_i$. Now, the constraint $z_i(x) \geq g_i$ enters into the budget $S$ and $P_i$ does not participate in the definition of $D_i$.

He is “out of the game” because he will surely realize at least $g_i$. The strategies for the following steps is left to $k-1$ players.

Any choice that $P_i$ makes has to be approved by other players. Thus, the strategy can be realized only if the players cooperate or by applying the rules of the regulatory subject, if such one exists. For better understanding we provide the following example.

Example 2. Let $k$ and $\lambda_i, i = 1, 2, \ldots, k$ be as in example 1. Suppose that at least one player (say $P_1$) is not satisfied with the proposed solution. Which strategy should be chosen?

As we have seen, to make any improvements, $P_2$ must decrease his aspiration level or set the absolute level below $\lambda d_2 = 0.6d_2$. Since $\lambda_2 > 1$, $P_3$ may turn his aspiration level into the absolute one. $P_1$ and $P_4$ may do nothing or they may even increase their aspiration levels. Thus, the necessary condition for the improvement is that $P_2$ should make the right move. If he does not agree, then the improvement is impossible. However, if the regulatory subject is present, $P_2$ may be forced to comply. The question is: To what extent does $d_2$ have to be decreased? Could a more specific rule with a mathematical formulation be given? The answer is affirmative but, if we do so, it must be part of the cooperation strategy among the players. Namely, the new level $d_2$ has to satisfy $d_2 < d_2$. This is the necessary condition for improving the solution. Since $\lambda_1 = 0.75$ is the closest indicator to $\lambda_2 = 0.6 = \lambda$, we can require

$$d_2^* \leq \frac{\lambda_2}{\lambda_1} d_2 = \frac{0.6}{0.75} d_2 = 0.8d_2.$$

Then, in the next step we shall have $\lambda \geq 0.75$.

Generally, if we want to have $\lambda \geq \mu$ in the next step, we have to require

$$d_i^* \leq \frac{\lambda_i}{\mu} d_i, \quad i = 1, 2, \ldots, k. \quad (8)$$

Thus, the players with $\lambda_i < \mu$ have to decrease their levels and those with $\lambda_i \geq \mu$ have a possibility to increase them. After this step is completed we can define $v > \mu$ for the next step, etc. The request (8) may be made by a regulatory subject, if present. Relying on (8) it can “lead the game” towards the end (equilibrium) through a given number of iterations ($s \geq 2$) by defining $\mu_2 < \mu_3 < \cdots < \mu_i$. If we want to apply (8) in the “game strategy”, we can drop the main cooperation principle contained in the method (unless it was a part of it). Still, it is not necessary to apply (8). The principle of cooperation is enough because we already have a powerful mechanism with which to control the moves of the players: the indicators! They clearly detect any wrong move. If the aspiration level is not decreased enough, then the indicator remains the optimal one (the smallest one). If it is decreased too much, than the indicator becomes large, which signal is to the player that he should increase the level. Thus, nothing can be lost and no one can go beyond the given possibilities. If, for some reasons, cooperation is not possible then (8) could be applied.

The final question is: When is the game over? Can we define a stopping criterion? A natural answer offers itself: the game could be over when there is no indicator less than one. This means that none of the players attains less than he has required. But it need not be the rule. The game may end before or after this happens if all the players agree to that. We can see that our method is very flexible. It allows us to define the number of iterations or/and stopping criterion or to leave the final decision to the players or to the regulatory subject.

Finally, we give the main conclusions, which could be used as recommendations and principles for creating the game strategy. Suppose that you are the player $P_i, i \in \{1, 2, \ldots, k\}$. How can you contribute to building the right strategy?

1. If your indicator is the optimal one ($\lambda_i = \lambda$) then it means that your demand is too high relative to the others. If someone is not satisfied, that is because you have limited him. If you want to change the situation you have to reduce your demand.
2. Increasing your demand by any amount would disable the others to receive more.
3. Decreasing your demand by any amount would enable the others to receive more.
4. If the others, particularly those whose indicators are not greater than yours, reduce their demands then you would expect to receive more.
5. If the others increase their demands then you would expect to receive less.
6. A reduction of your indicator in the subsequent steps shows that your demands are too sharp.

We clearly see that the benefits of the other players depend on your decision and vice versa. This is the inherent property of this method. We can again recognize an ancient wisdom here: as you give so you shall receive.

These principles may be established as the rules of the method. By following these rules the players will surely reach the desired equilibrium.

3.3. Nonlinear case

Our method could be easily extended to the nonlinear case, which was also noticed and suggested by one anonymous referee. Namely, Theorem 1 holds in a very general case, for an arbitrarily closed and bounded (not necessarily convex) set $S$. It means that the functions which define the boundary of $S$ (constraints) can be any continuous functions. The same goes for the objective functions $z_i(x), i = 1, 2, \ldots, k$ which have to be defined on $S (S \subseteq \text{domain}(z_i))$.

Note that, in the light of Theorem 1, problem (5) can be also stated in the general simple form,

$$\max\{\lambda : \Gamma_1 \neq \emptyset\}.$$  \hspace{1cm} (9)

Thus, if the assumptions of Theorem 1 hold ($D \neq \emptyset$ and $\Gamma_0 \neq \emptyset$) then (9) has the solution. All other conclusions concerning optimal points, indicators and possible iterations remain the same. But since (9), i.e. (5), is not LPP any more, serious computational problems may occur. The methods of analytic geometry may help, especially if problems have two or three variables.

As an illustration we provide the following example.

Example 3 (linear objective functions, nonlinearity in $S$). Let us solve the problem

$$\max_{x \in S} (x_1 + x_2), \text{ where } S = \{(x_1, x_2) : x_1, x_2 \geq 0, x_1^2 + x_2^2 \leq 400\},$$

with aspiration levels $d_1 = 7, d_2 = 4$.

We have $D_1 = \{(x_1, x_2) : x_1, x_2 \geq 0, x_1 + x_2 \geq 7\lambda, x_1, x_2 \geq 4\lambda\}, D = D_1$. Using the geometric representation we easily obtain $x_1 = 12, x_2 = 16, \lambda = 4$, where both constraints are active ($\lambda_1 = \lambda_2 = \lambda$). If we change the level $d_2 = 4$ to $d_2 = 3$ then we obtain $x_1 = x_2 = 10/\sqrt{2}, \lambda = (20/\sqrt{2})/7 \approx 4.041$. Here $x_1 + x_2 = 7\lambda$ is the active constraint and $x_1 = 10/\sqrt{2} > 2\lambda = (60/\sqrt{2})/7$ is the passive one ($\lambda_1 = \lambda, \lambda_2 = 10/\sqrt{2}/3 \approx 4.714 > \lambda$), etc. Thus, the method is applicable without modifications.

If the objective functions $z_i(x), i = 1, 2, \ldots, k$ are nonlinear then the transition $D \rightarrow D_1$ is generally not a homothetic transformation. However, it does not matter. It is essential that the objective functions are continuous, which ensures the continuous dependence of $D_1$ (and $\Gamma_1$) on $\lambda$. We also provide an example for illustration.

Example 4 (nonlinear objective functions, linearity in $S$). For given $a, b, c > 0$ let us solve the problem

$$\max_{x \in S} (a x_1 - b x_2), \text{ where } S = \{(x_1, x_2) : x_1, x_2 \geq 0, x_1 + x_2 \leq c\},$$

and $d_1 = a, d_2 = b$.

We have $D_1 = \{(x_1, x_2) : x_2 \geq \lambda a, \ln x_1 - x_2 \geq \lambda b\}, D = D_1$ (use the geometric representation again). Note that $D_1 \neq \emptyset$ (and $D \neq \emptyset$) for any choice of $a, b > 0$ and $\lambda > 0$ but for $c < 1$ the problem has no solution because $\Gamma_1 = S \cap D_1 = \emptyset$ (the assumptions of Theorem 1 do not hold because $\Gamma_0 = \emptyset$). For $c \geq 1$ the unique solution is given by $x_2 = \lambda a, x_1 = e^{(a-b)/\lambda}$, where $\lambda$ is the solution of equation $\lambda a + e^{(a-b)\lambda} = c$, which obviously requires an iterative method to be solved (except for $c = 1$ when $x_1 = 1, x_2 = 0, \lambda = 0$).

In conclusion, the application of our method to a nonlinear case is possible but it requires that computational methods be further researched and developed (probably separately for different kinds of nonlinearity, e.g. quadratic, cubic, fractional, etc.). This kind of research may be presented in subsequent papers.

Table 1

Data for Example 5.

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<th>Product 2 ($x_2$ units)</th>
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4. Numerical examples

Let us apply the above method to the following manufacturing problem. The simplified version of this problem was studied in [6].

Example 5. A manufacturing company has six machine types – milling machine, lathe, grinder, jig, saw, drill press, and band saw – where capacities are to be devoted to produce three products: 1, 2 and 3. Decision makers have the four objectives of maximizing profits, quality, worker satisfaction, and total revenue. The data is given in the Table 1.

The associated MOLPP (1) is as follows. We see that $k = 4$ and $n = 3$. Thus, we have four decision makers (players) $P_i, i = 1, 2, 3, 4$ with their objective functions $z_i(x), x \in \mathbb{R}^3$. We have the following problem

$$\max_{x \in S}(z_1(x), z_2(x), z_3(x), z_4(x)), x = (x_1, x_2, x_3), \text{ where } z_1 = 50x_1 + 100x_2 + 17.5x_3, \quad z_2 = 50x_1 + 50x_2 + 100x_3,$$

$$z_3 = 20x_1 + 50x_2 + 100x_3, \quad z_4 = 25x_1 + 75x_2 + 12x_3, \quad \text{and } S = \left\{ \begin{array}{l}
12x_1 + 17x_2 \leq 1400 \\
3x_1 + 9x_2 + 8x_3 \leq 1000 \\
10x_1 + 13x_2 + 15x_3 \leq 1750 \\
6x_1 + 16x_2 \leq 1325 \\
12x_2 + 7x_3 \leq 900 \\
9.5x_1 + 9.5x_2 + 4x_3 \leq 1075 \\
x_1, x_2, x_3 \geq 0
\end{array} \right\}. \tag{10}$$

We have four LP problems (2) which are assigned to (10),

$$\max_{x \in S} z_i(x), \quad i = 1, 2, 3, 4.$$

Lindo 6.1 software was used to solve the problems and the following solutions were obtained:

$$\hat{z}_1 = 8041.14, \quad x^{(1)} = (44.94, 50.63, 41.77),$$

$$\hat{z}_2 = 10137.99, \quad x^{(2)} = (22.28, 31.57, 74.46),$$

$$\hat{z}_3 = 9615.88, \quad x^{(3)} = (0.26, 69.82, 81.91),$$

$$\hat{z}_4 = 5885.42, \quad x^{(1)} = (10.42, 75.0). \tag{11}$$

Now we can apply our method.

The first step. Suppose that the players, taking into account the obtained results (11), set their aspiration levels at maximal values,

$$d_1 = 8041.14, \quad d_2 = 10137.99, \quad d_3 = 9615.88, \quad d_4 = 5885.42.$$

Since these levels are achieved at different optimal points, the players know that they cannot realize them to the full extent. But, to what extent could their levels be realized? To answer this question we state LPP (5) which is assigned to (10),

$$\max_{(x, \lambda) \in C} \lambda, \text{ where } G = \left\{ (x, \lambda) \in \mathbb{R}^3 : x \in S, \lambda \geq 0, z_1(x) \geq 8041.14\lambda, z_2(x) \geq 10137.99\lambda, z_3(x) \geq 9615.88\lambda, z_4(x) \geq 5885.42\lambda \right\}. \tag{12}$$

Using the same software we obtained the following solution:

$$\hat{x} = (39.63, 45.21, 51.07), \quad \hat{\lambda} = 0.8486, \quad z_1(\hat{x}) = 7396.23, \quad z_2(\hat{x}) = 9349.00, \quad z_3(\hat{x}) = 8160.10, \quad z_4(\hat{x}) = 4994.34.$$

After computing the indicators (6) we have obtained

$$\lambda_1 = 0.9198, \quad \lambda_2 = 0.9222, \quad \lambda_3 = \lambda_4 = 0.8486 = \hat{\lambda}.$$

We see that, on the given budget $S$, the players may realize at least 84.86% of their aspirations ($P_3$ and $P_4$ exactly 84.86% while $P_1$ and $P_2$ may realize even more, 91.98% and 92.22%, respectively). Evidently, the constraints for $P_3$ and $P_4$ are active and those for $P_1$ and $P_2$ are the passive ones. This means that the aspiration levels of $P_3$ and $P_4$ are set too high when compared to those of $P_1$ and $P_2$ (see examples 1 and 2 above). If the players are satisfied with the achieved results then the obtained solution is the final one. If they are not then we perform the next step.

The second step. The initial data for this step have to be defined. The obtained indicators will help in making decisions. We see that $P_1$ and $P_2$ can increase their aspiration levels up to (see (7))

$$d_1 \leq \frac{\lambda_1}{\hat{\lambda}} d_3 = 8715.81, \quad d_2 \leq \frac{\lambda_2}{\hat{\lambda}} d_2 = 11017.27,$$

which will not change the solution. However, any increase of $d_3$ or $d_4$ will decrease $\hat{\lambda}$. Thus, to make any improvement, $P_3$ or $P_4$ (or both of them) has to decrease his aspiration level. $P_1$ and $P_2$ have several choices: they need not to change anything or they may somewhat increase their demands (which is useless here because the levels were already set to the maximal val-
ues) or they may fix the absolute levels (not greater than the realized ones). Suppose that the players have reached the following agreement. $P_1$ will not change his aspiration level. $P_2$ will set the absolute level $z_2(x) \geq 9000$. $P_3$ and $P_4$ will decrease their aspiration levels to 8000 and 5000, respectively. Now, we need to solve the problem

$$\max_{(x, \lambda) \in G} \lambda,$$

where

$$G = \{(x, \lambda) \in \mathbb{R}^d : x \in S, \lambda \geq 0, z_1(x) \geq 8041.14\lambda, z_2(x) \geq 9000, z_3(x) \geq 8000\lambda, z_4(x) \geq 5000\lambda\}.$$

Note that $P_2$ is "out of the game". He will surely realize his aspiration. Any further decision is left to three players, $P_1$, $P_3$ and $P_4$. The solution is

$$\hat{x} = (44.08, 48.72, 45.06), \quad \hat{\lambda} = 0.9779, \quad z_1(\hat{x}) = 7863.55, \quad z_2(\hat{x}) = 9145.50, \quad z_3(\hat{x}) = 7823.10, \quad z_4(\hat{x}) = 5295.97.$$

We again compute the associated indicators (6). Note that now we have three indicators,

$$\lambda_1 = \lambda_2 = 0.9779 = \hat{\lambda}, \quad \lambda_4 = 1.0592.$$

We see that $P_1$ and $P_3$ have active constraints while $P_4$ has the passive one. Note also that $P_4$ realizes his aspiration level to the full extent ($\lambda_4 > 1$). Since the players realized their demands to the very high extent (97.99% or more), the story may be ended. But, it also may be continued if the players want so. Let us suppose the last is the case.

The third step. Suppose that we have the following agreement. Since $P_1$ has not changed his aspiration level during the decision-making process so far, he will decrease his aspiration level to 7500. This will enable $P_3$ to realize more (he need not to change anything). $P_4$ will turn his aspiration level of 5000 into the absolute one. Now the final decision-making process is left to $P_1$ and $P_3$ while $P_2$ and $P_4$ are "out of the game". Thus, we have the problem

$$\max_{(x, \lambda) \in G} \lambda,$$

where

$$G = \{(x, \lambda) \in \mathbb{R}^d : x \in S, \lambda \geq 0, z_1(x) \geq 7500\lambda, z_2(x) \geq 9000, z_3(x) \geq 8000\lambda, z_4(x) \geq 5000\}.$$

We have obtained

$$\hat{x} = (41.07, 46.35, 49.12), \quad \hat{\lambda} = 1.0064, \quad z_1(\hat{x}) = 7548.10, \quad z_2(\hat{x}) = 9283.00, \quad z_3(\hat{x}) = 8050.90, \quad z_4(\hat{x}) = 5092.44.$$

The remaining two indicators are the optimal ones, $\lambda_1 = \lambda_3 = \hat{\lambda} = 1.0064$. Since $\hat{\lambda} > 1$, the players have fully realized their demands (even a little more than 100%). Thus, "the game is over". Even now, if the players want to change the solution for some reason, the method allows them to proceed further.

5. Conclusions

In the paper we have introduced a new, efficient method for solving linear multiobjective optimization problems. The main properties and advantages of the method are summarized below.

The method is iterative. It includes the basic step which may be repeated until the satisfactory equilibrium is attained. Numerically, this step is very simple. It consists in solving a single linear optimization problem, which can be done by using any appropriate software. The obtained unique solution also yields the objective indicators. They show which aspirations are set higher or lower in relation to the others. This can be a guideline in defining the strategy for the next step. The indicators clearly show the moves which need to be made to come closer to the desired equilibrium state. Thus, at each stage of the process we are able to understand why we have obtained such a solution and we also see what should be done to drive the solution in the desired direction. The method also allows that a quite different strategy be applied if necessary. As a result, the participants (players) can adjust their aspirations until they reach the state of equilibrium which is satisfactory to everyone. The method also enables the players to detect if a such state does not exist.

References


