Multiple orthogonal polynomials on the semicircle and applications

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**A R T I C L E I N F O**

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**A B S T R A C T**

In this paper two types of multiple orthogonal polynomials on the semicircle with respect to a set of \( r \) different weight functions are defined. Such polynomials are generalizations of polynomials orthogonal on the semicircle with respect to a complex-valued inner product \( \langle f, g \rangle = \int_0^\pi f(e^{i\theta})g(e^{i\theta})w(e^{i\theta})\, d\theta \). The existence and uniqueness of introduced multiple orthogonal polynomials for certain classes of weight functions are proved. Some properties of multiple orthogonal polynomials on the semicircle including certain recurrence relations of order \( r + 1 \) are presented. Finally, an application in numerical integration is given.

**1. Introduction**

In the theory of orthogonal polynomials it is well known that a positive definite inner product generates a unique set of real orthogonal polynomials (see e.g., [6,10]). When inner product is not Hermitian, the existence of the corresponding orthogonal polynomials is not guaranteed. Gautschi and Milovanović in [8] introduced polynomials orthogonal on the semicircle with respect to the following not Hermitian inner product:

\[
\left< f, g \right> = \int_0^\pi f(e^{i\theta})g(e^{i\theta})w(e^{i\theta})\, d\theta.
\]

They proved existence and uniqueness of the corresponding orthogonal polynomials, discussed their properties and applications involving quadrature rules of Gaussian type over the semicircle, numerical differentiation, and the computation of Cauchy principal value integrals. Generalizing that work, Gautschi, Landau, and Milovanović in [7] considered orthogonality on the semicircle with respect to the suitable complex "weight function".

A generalization of orthogonal polynomials in the sense that they satisfy \( r \in \mathbb{N} \) orthogonality conditions leads to the concept of multiple orthogonal polynomials. Multiple orthogonal polynomials arise naturally in the theory of simultaneous rational approximation, in particular in Hermite–Padé approximation of a system of \( r \) (Markov) functions. For more details about multiple orthogonal polynomials, we refer to the book by Nikishin and Sorokin [17, Chapter 4], surveys by Aptekarev [1], de Bruin [4], and Milovanović and Stanić [16], as well as the papers by Piñeiro [18], Sorokin [19–21], Van Assche [22], Van Assche and Coussement [25], Aptekarev, Branquinho, and Van Assche [2], and Chapter 23 of Ismail’s book [9].

A generalization of orthogonal polynomials on the semicircle in the sense that they satisfy \( r \in \mathbb{N} \) orthogonality conditions was investigated by Milovanović and Stanić in [15] (see also [13,16]), where only one type of multiple orthogonal polynomials on the semicircle with respect to so called nearly diagonal multi–index was considered. In this paper we introduce two types of multiple orthogonal polynomials on the semicircle with respect to arbitrary multi–index and prove their main...
properties. For that purpose we repeat some basic facts about polynomials orthogonal on the semicircle in Section 2 and about multiple orthogonal (real) polynomials in Section 3. Finally, in Section 4 the concept of multiple orthogonality is transferred to the semicircle, where two types of polynomials orthogonal on the semicircle are defined and their main properties are proved. Section 5 is devoted to recurrence relations. Finally, in Section 6 we present an application of introduced multiple orthogonal systems in numerical integration.

2. Polynomials orthogonal on the semicircle

For nonnegative integer $n$ by $\mathcal{P}_n$ we denote the set of all polynomials of degree at most $n$, and by $\mathcal{P}$ the set of all polynomials. Let $w$ be a weight function, which is positive and integrable on the open interval $(-1, 1)$, though possibly singular at the endpoints, and which can be extended to a function $w(z)$ holomorphic in the half disc $D_+ = \{z \in \mathbb{C} : |z| < 1, \text{Im} z > 0\}$. Let us considered the following two inner products:

$$\langle f, g \rangle = \int_{-1}^{1} f(x) \overline{g(x)} w(x) \, dx. \quad (2.1)$$

$$\langle f, g \rangle = \int_{\Gamma} f(z) g(z) (iz)^{-1} w(z) \, dz = \int_{0}^{\pi} f(e^{i\theta}) g(e^{i\theta}) w(e^{i\theta}) \, d\theta, \quad (2.2)$$

where $\Gamma$ is the circular part of $\partial D_+$ and all integrals are assumed to exist, possibly as appropriately defined improper integrals.

The inner product (2.1) is positive definite and, therefore, generates a unique set of real orthogonal polynomials $\{p_n\}$ ($p_k$ is monic polynomial of degree $k$). The inner product (2.2) is not Hermitian and the existence of the corresponding orthogonal polynomials, therefore, is not guaranteed.

**Definition 2.1.** A system of monic complex polynomials $\{\pi_k\}$ ($\pi_k$ is of degree $k$) is called orthogonal on the semicircle if $[\pi_k, \pi_{k+1}] = 0$ for $k \neq \ell$ and $[\pi_k, \pi_k] \neq 0$ for $k = \ell, \ell - 1, \ell - 2, \ldots$.

We denote by $m_k$ and $\mu_k$ the moments associated with the inner products (2.1) and (2.2), respectively,

$$m_k = (x^k, 1), \quad \mu_k = (\mu^k, 1), \quad k = 0, 1, 2, \ldots \quad (2.3)$$

Gautschi, Landau, and Milovanović in [7] have established the existence of orthogonal polynomials $\{\pi_k\}$ assuming only that

$$\text{Re} \mu_0 = \text{Re} [1, 1] = \text{Re} \int_{0}^{\pi} w(e^{i\theta}) \, d\theta \neq 0.$$ 

Let $C_\varepsilon$, $\varepsilon > 0$, denotes the boundary of $D_+$ with small circular parts $c_{\varepsilon, \pm 1}$ of radii $\varepsilon$ and centers at $\pm 1$ spared out. We assume that $w$ is such that

$$\lim_{\varepsilon \to 0} \int_{c_{\varepsilon, 1}} g(z) w(z) \, dz = 0, \quad \text{for all } g \in \mathcal{P}.$$ 

Then the following equality

$$0 = \int_{C} g(z) w(z) \, dz = \int_{\Gamma} g(z) w(z) \, dz + \int_{-1}^{1} g(x) w(x) \, dx,$$

holds for all $g \in \mathcal{P}$ (see [7]).

The monic (real) polynomials $\{p_k(z)\}$, orthogonal with respect to the inner product (2.1), as well as the associated polynomials of the second kind,

$$q_k(z) = \int_{-1}^{1} \frac{p_k(x) - p_k(x)}{z - x} w(x) \, dx, \quad k = 0, 1, 2, \ldots,$$

are known to satisfy a three–term recurrence relation of the form

$$y_{k+1} = (z - a_k)y_k - b_k y_{k-1}, \quad k = 0, 1, 2, \ldots, \quad (2.4)$$

with initial conditions $y_{-1} = 0$, $y_0 = 1$ for $\{p_k\}$, and $y_{-1} = -1$, $y_0 = 0$ for $\{q_k\}$. Recurrence coefficient $b_0$ could be chosen arbitrary, but we assume that $b_0 = m_0$.

Gautschi, Landau, and Milovanović [7, Theorem 2.1] proved the existence of a unique system of monic (complex) orthogonal polynomials $\{\pi_k\}$ with respect to the inner product (2.2) and represented $\pi_n$ as a linear complex combination of $p_n$ and $p_{n-1}$:

$$\pi_n(z) = p_n(z) - i \bar{b}_n \bar{p}_{n-1}(z), \quad n = 0, 1, 2, \ldots \quad (2.4)$$

$$\text{Re} \mu_0 = \text{Re} [1, 1] = \text{Re} \int_{0}^{\pi} w(e^{i\theta}) \, d\theta \neq 0.$$
where
\[ \theta_{n-1} = \frac{\mu_n p_n(0) + i q_n(0)}{i \mu_0 p_{n-1}(0) - q_{n-1}(0)}, \quad n = 0, 1, 2, \ldots, \]
or, alternatively,
\[ \theta_n = i a_n + \frac{b_n}{\theta_{n-1}}, \quad n = 0, 1, 2, \ldots; \quad \theta_{-1} = \mu_0, \]
where \( a_n, b_n \) are the recurrence coefficients in (2.4) and \( \mu_0 = [1, 1] \) the corresponding zero moment given by (2.3).

Under certain conditions the zeros of polynomials orthogonal on the semicircle lie in \( D_+ \) (see \([8, 7, 11, 12]\)).

### 3. Multiple orthogonal polynomials

Let \( r \) be a positive integer and let \( w_1, w_2, \ldots, w_r \) be \( r \) weight functions on the real line such that the support of each \( w_i \) is a subset of an interval \( E_i \). Let \( \mathbf{n} = (n_1, n_2, \ldots, n_r) \) be a multi-index, i.e., a vector of \( r \) nonnegative integers, with length \( |\mathbf{n}| = n_1 + n_2 + \cdots + n_r \).

We introduce a partial order on multi–indices in the following way:
\[ \mathbf{m} \preceq \mathbf{n} \iff m_v \leq n_v \quad \text{for every} \quad v = 1, 2, \ldots, r. \]

There are two types of multiple orthogonal polynomials (see, e.g., \([25]\)).

**Definition 3.1.** Type I multiple orthogonal polynomials are collected in a vector \( (A_{n,1}, A_{n,2}, \ldots, A_{n,r}) \) of \( r \) polynomials such that \( A_{n,v} \) has degree \( n_v - 1 \) and the following orthogonality conditions hold:
\[
\sum_{i=1}^{r} \int_{E_i} A_{n,v} x^k w_i(x) \, dx = 0, \quad k = 0, 1, 2, \ldots, |\mathbf{n}| - 2,
\]
with the normalization
\[
\sum_{i=1}^{r} \int_{E_i} A_{n,v} x^{n_v-1} w_i(x) \, dx = 1.
\]

**Definition 3.2.** Type II multiple orthogonal polynomial is a monic polynomial \( P_n \) of degree \( |\mathbf{n}| \) which satisfies the following orthogonality conditions:
\[
\int_{E_v} P_n(x) x^k w_v(x) \, dx = 0, \quad k = 0, 1, \ldots, n_v - 1, \quad v = 1, 2, \ldots, r.
\]

The orthogonality conditions for type II multiple orthogonal polynomials give \( |\mathbf{n}| \) linear equations for the \( |\mathbf{n}| \) unknown coefficients \( a_{\mathbf{n}} \) of the polynomial \( P_n(x) = \sum_{k=0}^{|\mathbf{n}|} a_{\mathbf{n},k} x^k \), where \( a_{\mathbf{n},|\mathbf{n}|} = 1 \). Since the matrix of coefficients of this system can be singular, we need some additional conditions on the \( r \) weight functions to provide the uniqueness of the multiple orthogonal polynomial.

The type II multiple orthogonal polynomial \( P_n(x) \) is unique if and only if vector of the type I multiple orthogonal polynomials is unique (see \([9]\)). If the polynomial \( P_n(x) \) is unique, then the multi–index \( \mathbf{n} \) is normal.

There are two distinct cases for which all the multi–indices are normal for multiple orthogonal polynomials (see \([25]\)):

1. Angelesco systems, where the intervals \( E_i \), on which the weight functions are supported, are disjoint, i.e., \( E_i \cap E_j = \emptyset \) for \( 1 \leq i \neq j \leq r \).
2. AT systems for the multi–index \( \mathbf{n} \), where all weight functions are supported on the same interval \( E \) and the set
\[ \{ x^k w_v(x) : k = 0, 1, \ldots, n_v - 1, \quad v = 1, 2, \ldots, r \} \]
form a Chebyshev system on \( E \).

The following two theorems have been proved in \([25]\).

**Theorem 3.1.** In an Angelesco system the type II multiple orthogonal polynomial \( P_n(x) \) factors into \( r \) polynomials \( \prod_{v=1}^{r} q_{n_v}(x) \), where each \( q_{n_v} \) has exactly \( n_v \) zeros on \( E_v \).

**Theorem 3.2.** In an AT system the type II multiple orthogonal polynomial \( P_n(x) \) has exactly \( |\mathbf{n}| \) zeros on \( E \). For the type I vector of multiple orthogonal polynomials, the linear combination \( \sum_{v=1}^{r} A_{n,v}(x) w_v(x) \) has exactly \( |\mathbf{n}| - 1 \) zeros on \( E \).
Multiple orthogonal polynomials satisfy recurrence relation of order \( r + 1 \). First it was proved for type II for so called nearly diagonal multi–indices [23]. Let \( n \) be a nonnegative integer, written as \( n = \ell r + j \), with \( \ell = \lfloor n/r \rfloor \) and \( 0 \leq j < r \) (\( \lfloor t \rfloor \) denotes the integer part of \( t \)). The nearly diagonal multi–index \( \mathbf{d}(n) \) corresponding to \( n \) is given by

\[
d(n) = (\ell + 1, \ell + 1, \ldots, \ell + 1, \ell, \ell, \ldots, \ell).
\]

A numerical procedure for construction of type II multiple orthogonal polynomials for nearly diagonal multi–indices, based on the discretized Stieltjes–Gautschi procedure (cf. [5]), was given in [14].

Recurrence relations for general multi–indices, only involving nearest neighbor multi–indices \( \mathbf{n} = \mathbf{e}_v \), where \( \mathbf{e}_v \) are the standard unit vectors with 1 on the \( v \)th entry, were given in [9,24]. Here we give a special case of those recurrence relations, which we need in the sequel.

By \( \mathbf{0} \) we denote zero multi–index, \( \mathbf{0} = (0,0,\ldots,0) \). Let \( (i_1,i_2,\ldots,i_v) \) be permutation of \( (1,2,\ldots,r) \) and let \( \sigma_v \) be the following multi–indices

\[
\sigma_v = \sum_{j=1}^v \mathbf{e}_{i_j}, \quad v = 1,2,\ldots,r.
\]

Thus, \( \sigma_v \) has \( v \) entries 1 and \( r - v \) entries 0. Then \( P_{\mathbf{n} - \sigma_v} \) is polynomial of degree \( |\mathbf{n}| - v, \quad v = 1,2,\ldots,r, \) and \( \mathbf{n} - \sigma_v = (n_1-1,n_2-1,\ldots,n_r-1) \). Choose \( k \in \{1,2,\ldots,r\} \) and suppose that all multi–indices \( \mathbf{m} \leq \mathbf{n} + \mathbf{e}_k \) are normal. In [9] it was proved that the following recurrence relation

\[
x P_{\mathbf{n}}(x) = P_{\mathbf{n} - \sigma_v}(x) + a_{v0}(k)P_{\mathbf{n}}(x) + \sum_{v=1}^r a_{v}(\mathbf{n})P_{\mathbf{n} - \sigma_v}(x),
\]

holds, where \( a_{v0}(k) \) and \( a_{v}(\mathbf{n}) \) are real numbers.

**Remark 3.1.** If \( (1,1,\ldots,1) \not\leq \mathbf{n} \), then \( \mathbf{n} - \sigma_v \) is also a multi–index. But, the problem appears when multi–index \( \mathbf{n} \) has some entries equal to 0, i.e., when \( \mathbf{n} - \sigma_v \) has negative entries. Also, since (3.3) is recurrence relation of order \( r + 1 \), we need \( r + 1 \) initial conditions. The problem of the initial conditions is closely related to the problem when \( \mathbf{n} \) has some entries equal to 0. But, if \( n_v = 0 \), for some \( \ell \in \{1,2,\ldots,r\} \), then in fact we do not have orthogonality with respect to weight function \( w_v \), i.e., we have orthogonality with respect to \( r - 1 \) weight functions; if \( \mathbf{n} \) has two entries equal to 0, we have orthogonality with respect to \( r - 2 \) weight functions, and so on. So, if \( (1,1,\ldots,1) \not\leq \mathbf{n} \) and \( n_1 = \cdots = n_p = 0 \) for certain \( p \in \{1,2,\ldots,r\} \), we take permutation \( (i_1,i_2,\ldots,i_v) \) with \( \{i_1,i_2,\ldots,i_{p-1}\} = \{1,\ldots,p\} \), and set \( P_{\mathbf{n} - \sigma_v}(x) = 0 \) for \( j = r, r - 1,\ldots,r - p + 1, \) i.e., \( P_{\mathbf{n}}(x) = 0 \) whenever multi–index \( \mathbf{m} \) has at least one negative entry. Now, it is easy to see that the initial conditions for recurrence relation (3.3) are given by \( P_0(x) = 1 \), and \( P_{-\sigma_v}(x) = 0 \) for \( j = 1,2,\ldots,r \).

For \( r = 1 \), type II multiple orthogonal polynomials reduce to ordinary orthogonal polynomials and recurrence relation of order \( r + 1 \) reduces to the well known three–term recurrence relation.

### 4. Multiple orthogonal polynomials on the semicircle

**Definition 4.1.** Let \( \mathbf{n} = (n_1,n_2,\ldots,n_r) \) be a multi–index (\( r \) is a positive integer) and let \( W = \{w_1,w_2,\ldots,w_r\} \) be an AT system of weight functions for the multi–index \( \mathbf{n} \), such that each \( w_v, \quad v = 1,2,\ldots,r, \) is positive and integrable on the open interval \((-1,1)\), though possibly singular at the endpoints, and which can be extended to a function \( w_v(z) \) holomorphic in the half disc \( D_v \). If, in addition, each weight function \( w_v, \quad v = 1,2,\ldots,r, \) satisfies \( \text{Re} \mu_v(0) \neq 0 \) and

\[
\lim_{\varepsilon \downarrow 0} \int_{\varepsilon c_{v+1}} g(z)w_v(z)dz = 0, \quad \text{for all} \quad g \in \mathcal{P},
\]

then such system \( W \) is admissible system for the multi–index \( \mathbf{n} \).

It is easy to see that for an admissible system \( W \) the following equations

\[
0 = \int_{-1}^{1} g(z)w_v(z)dz + \int_{-1}^{1} g(x)w_v(x)dx, \quad v = 1,2,\ldots,r,
\]

\[
\int_{-1}^{1} \frac{g(z)w_v(z)}{iz} dz = \pi g(0)w_v(0) + i \int_{-1}^{1} \frac{g(x)w_v(x)}{x} dx, \quad v = 1,2,\ldots,r,
\]

hold for all polynomials \( g \).

Let us introduce the following notations for the inner products

\[
(f,g)_v = \int_{-1}^{1} f(x)\overline{g(x)}w_v(x)dx,
\]

(4.3)
\[ f, g \rangle_{\Gamma} = \int_{\Gamma} f(z)g(z)(iz)^{-1}w_{\nu}(z)\,dz = \int_{0}^{\pi} f(e^{i\theta})g(e^{i\theta})w_{\nu}(e^{i\theta})\,d\theta, \quad (4.4) \]

for \( v = 1, 2, \ldots, r \), and for the corresponding moments:

\[ m_{k}^{(v)} = (x^k, 1)_v, \quad \mu_k^{(v)} = [x^k, 1]_v, \quad k \geq 0, \quad v = 1, 2, \ldots, r. \quad (4.5) \]

For zero moments \( \mu_0^{(v)} \) from (4.2) we have

\[ \mu_0^{(v)} = \int (iz)^{-1}w_{\nu}(z)\,dz = \pi w_{\nu}(0) + i \int_{-1}^{1} \frac{w_{\nu}(x)}{x} \,dx, \quad v = 1, 2, \ldots, r. \quad (4.6) \]

We define two types of multiple orthogonal polynomials on the semicircle.

**Definition 4.2.** Type I multiple orthogonal polynomials on the semicircle are collected in a vector \( (B_{n,1}, B_{n,2}, \ldots, B_{n,r}) \) of \( r \) polynomials such that \( B_{n,v} \) has degree \( n_v - 1 \) and the following orthogonality conditions hold:

\[ \sum_{v=1}^{r} \int_{\Gamma} B_{n,v}(z) z^{k}(iz)^{-1}w_{\nu}(z)\,dz = 0, \quad k = 0, 1, 2, \ldots, |n| - 2, \quad (4.7) \]

with normalization

\[ \sum_{v=1}^{r} \int_{\Gamma} B_{n,v}(z) z^{n_v-1}(iz)^{-1}w_{\nu}(z)\,dz = 1. \quad (4.8) \]

The conditions (4.7) and (4.8) give a linear system of \(|n|\) equations for the \(|n|\) unknown coefficients of the polynomials \( B_{n,v}, \quad v = 1, 2, \ldots, r \). A multi-index \( n \) is *normal* for type I if the system (4.7) and (4.8) has unique solution. For the type I multiple orthogonal polynomials on the semicircle we define the following function:

\[ B_{n}(z) = \sum_{v=1}^{r} B_{n,v}(z)w_{\nu}(z). \quad (4.9) \]

For \( n \neq 0 \) the orthogonality conditions (4.7) and the normalizations (4.8) become

\[ \int_{\Gamma} B_{n}(x) z^{k}(iz)^{-1}\,dz = 0, \quad k = 0, 1, 2, \ldots, |n| - 2, \quad (4.10) \]

and

\[ \int_{\Gamma} B_{n}(x) z^{n-1}(iz)^{-1}\,dz = 1, \quad (4.11) \]

respectively.

**Remark 4.1.** If \( n_v = 0 \) for some \( v \in \{ 1, 2, \ldots, r \} \), then the corresponding type I coordinate is the zero polynomial, i.e., \( B_{n,v}(z) = 0 \). For \( n = 0 \) we have \( B_{0,v}(z) = 0, \quad v = 1, 2, \ldots, r \), and hence \( B_{0}(z) = 0 \). For \( n = e_{v}, \quad v = 1, 2, \ldots, r \), we have that \( B_{e,v} \) is a polynomial of degree 0, i.e., \( B_{e,v} = a_v \), where \( a_v \) is a constant which can be obtained from the normalization (4.8). Thus, \( \int_{\Gamma} a_v(iz)^{-1}w_{\nu}(z)\,dz = 1 \), i.e., \( a_v\mu_0^{(v)} = 1 \), where \( \mu_0^{(v)} \) is zero moment given by (4.6), and

\[ B_{e_v}(z) = B_{e,v}(z)w_{\nu}(z) = \left( \mu_0^{(v)} \right)^{-1}w_{\nu}(z), \quad v = 1, 2, \ldots, r. \quad (4.12) \]

Also, for \( (1, 1, \ldots, 1) \neq n \), we have \( B_{n,v}(z) = 0 \) if \( n_v = 0 \), and \( B_{n,v}(z) = B_{e,v}(z) \) if \( n_v = 1 \). Now, it is easy to get

\[ B_{e_{(i_1, i_2, \ldots, i_t)}}(z) = \sum_{j=1}^{p} B_{e_{(i_1, i_2, \ldots, i_t)}}^j(z)w_{j}(z) = \sum_{j=1}^{p} \left( \mu_0^{(j)} \right)^{-1}w_{j}(z), \quad v = 1, 2, \ldots, r, \quad (4.13) \]

where \( \sigma_j \) is given by (3.2) for the permutation \((i_1, i_2, \ldots, i_t)\).

**Definition 4.3.** Type II multiple orthogonal polynomial on the semicircle is a monic polynomial \( \Pi_n \) of degree \(|n|\) which satisfies the following orthogonality conditions:

\[ \int_{\Gamma} \Pi_n(z) z^{k}(iz)^{-1}w_{\nu}(z)\,dz = 0, \quad k = 0, 1, \ldots, n_v - 1, \quad v = 1, 2, \ldots, r. \quad (4.14) \]

**Remark 4.2.** If \( n_v = 0 \) for some \( v \in \{ 1, 2, \ldots, r \} \), then we do not have orthogonality conditions (4.14) with respect to weight \( w_v \).
The conditions (4.14) give a linear system of \( \lfloor n \rfloor \) equations for the unknown coefficients of the monic polynomial \( \Pi_n \). If this system has a unique solution, then \( n \) is a normal multi–index for type II.

**Lemma 4.1.** A multi–index \( n \) is normal for type I if and only if it is normal for type II.

**Proof.** The matrix of system (4.7) and (4.8) is given by

\[
M_n^I = \begin{bmatrix}
M_1 \quad M_2 \quad \cdots \quad M_r
\end{bmatrix}_{n \times n},
\]

where

\[
M_v = \begin{bmatrix}
\mu_1^{(v)} \quad \mu_2^{(v)} \quad \cdots \quad \mu_{n-1}^{(v)} \\
\mu_1^{(v)} \quad \mu_2^{(v)} \quad \cdots \quad \mu_{n-1}^{(v)} \\
\vdots \\
\mu_{n-1}^{(v)} \quad \mu_1^{(v)} \quad \cdots \quad \mu_{n-1}^{(v)}
\end{bmatrix}.
\]

The matrix of system (4.14) is given by

\[
M_n^II = \begin{bmatrix}
M_1^I \\
M_2^I \\
\vdots \\
M_r^I
\end{bmatrix}_{m \times n}.
\]

Obviously, \( M_{n}^{II} \) is the transpose of \( M_{n}^{I} \), which gives what is stated. \( \square \)

According to Lemma 4.1, in what follows we just talk on normal multi–indices. If all multi–indices are normal, then we have a perfect system.

**Remark 4.3.** Multiple orthogonal polynomials on the semicircle considered in [15,13,16] are in fact type II multiple orthogonal polynomials on the semicircle with respect to nearly diagonal multi–indices.

Let \( n \) be a multi–index and let \( W \) be an admissible system for all multi–indices \( m \leq n \). By \( P_m \), \( m \leq n \), we denote the corresponding type II multiple orthogonal (real) polynomials, which are uniquely determined due to properties of \( W \). Let us denote the associated polynomials of the second kind as follows:

\[
Q^{(v)}_{m}(z) = \int_{-1}^{1} \frac{P_m(z) - P_m(x)}{z - x} w_v(x) \, dx, \quad v = 1, 2, \ldots, r,
\]

(such polynomials satisfy the same recurrence relation as polynomials \( P_m \), but with different initial conditions), and

\[
D_n = \begin{bmatrix}
Q_{n-\sigma,0}^{(1)}(0) - i\mu_0^{(1)}P_{n-\sigma,0}(0) & \cdots & Q_{n-\sigma,\ell}^{(1)}(0) - i\mu_0^{(1)}P_{n-\sigma,\ell}(0) \\
Q_{n-\sigma,0}^{(2)}(0) - i\mu_0^{(2)}P_{n-\sigma,0}(0) & \cdots & Q_{n-\sigma,\ell}^{(2)}(0) - i\mu_0^{(2)}P_{n-\sigma,\ell}(0) \\
\vdots & & \vdots \\
Q_{n-\sigma,0}^{(p)}(0) - i\mu_0^{(p)}P_{n-\sigma,0}(0) & \cdots & Q_{n-\sigma,\ell}^{(p)}(0) - i\mu_0^{(p)}P_{n-\sigma,\ell}(0)
\end{bmatrix},
\]

where \( \sigma_i \) is given by (3.2), \( p \) is the number of zero entries in \( n \), and for \( p > 0 \sigma_i \) is chosen as it was explained in Remark 3.1.

**Theorem 4.1.** Let \( n \) be a multi–index and let \( W \) be an admissible system for all multi–indices \( m \leq n \), such that matrix \( D_n \) given by (4.17) is regular, where \( P_m \) are the corresponding type II orthogonal (real) polynomials and \( \sigma_j, j = 1, 2, \ldots, r \), are multi–indices given by (3.2). If \( n \) has entries equal to 0, permutation of indices for \( \sigma_i \) is chosen as it was explained in Remark 3.1. Then exists unique type II multiple orthogonal polynomial on the semicircle \( \Pi_n \) and the following representation holds

\[
\Pi_n(z) = P_n(z) + \sum_{j=1}^{r} \theta_{n,j}P_{n-\sigma_j}(z),
\]

where \( \theta_{n,j}, \ j = 1, 2, \ldots, r \), are constants.

**Proof.** Assume first that the type II multiple orthogonal polynomial on the semicircle \( \Pi_n \) exists. Let \( m_k, k = 0, 1, \ldots, \lfloor n \rfloor \), be multi–indices such that \( m_0 = 0 = (0, 0, \ldots, 0); \ |m_k| = k; \ m_{k+1} = m_k + e_r \) for some \( v \); \( m_k \leq n \) for all \( k \); and the last \( r + 1 \) multi–indices are given as follows.
\( \mathbf{m}_n = \mathbf{n}; \quad \mathbf{m}_{n-r} = \mathbf{n} - \mathbf{\sigma}_r, \quad v = 1, 2, \ldots, r. \)

The type II multiple orthogonal polynomials \( P_{m_k} \), \( k = 0, 1, \ldots, |\mathbf{n}| \), form a basis of the linear space \( P_{\mathbf{n}}. \)

Putting

\[
g(x) = -i\Pi_n(z) x^{k_1-1}, \quad 1 \leq k_i < n_i,
\]

in (4.1), for \( v = 1, 2, \ldots, r \), we get

\[
0 = \int_{\Gamma} \Pi_n(x) x^{k_1} (iz)^{-1} w_v(z) \, dz - i \int_{A} \Pi_n(x) x^{k_1-1} w_v(x) \, dx = [\Pi_n, x^{k_1-1}]_v - i (\Pi_n, x^{k_1-1})_v,
\]

hence, upon expanding \( \Pi_n \) in the polynomials \( \{P_{m_k}\}_{k=0}^{|\mathbf{n}|} \), because of orthogonality conditions (4.14) and (3.1) (since \( \mathbf{n} - \mathbf{\sigma}_r = (n_1 - 1, n_2 - 1, \ldots, n_r - 1) \), \( \mathbf{n} - \mathbf{\sigma}_r \leq \mathbf{n} - \mathbf{\sigma}_j \) for all \( j = 1, 2, \ldots, r - 1 \), and \( k_i - 1 < n_i - 1 \), we get representation (4.18) for some constants \( \rho_{n,j}, j = 1, 2, \ldots, r. \) To determine these constants, put

\[
g(x) = \frac{\Pi_n(z) - \Pi_n(0)}{iz} = -i \left( \frac{P_n(z) - P_n(0)}{z} + \sum_{j=1}^{r} \rho_{n,j} \frac{P_{n-\sigma_j}(z) - P_{n-\sigma_j}(0)}{z} \right),
\]

in (4.1) for \( v = 1, 2, \ldots, r \), and use the first expression for \( g \) to evaluate the first integral, and the second one to evaluate the second integral in (4.1). This for \( (1, 1, \ldots, 1) \preceq \mathbf{n} \) (then \( \Pi_{\mathbf{n}, 1} = 0 \) for all \( v = 1, \ldots, r \)) gives the following linear system

\[
\sum_{j=1}^{r} \rho_{n,j} \left( Q_n(\mathbf{\sigma}_j)(0) - i \mu_{n,j}(\mathbf{\sigma}_j)(0) \right) = i \mu_{n,1}(\mathbf{n})(0) - Q_n(\mathbf{n})(0), \quad v = 1, 2, \ldots, r.
\]

The matrix of the previous system is \( D_n \) (with \( p = 0 \)), hence it has unique solution.

Having in mind Remark 3.1 it is easy to get assertion in the case when \( (1, 1, \ldots, 1) \npreceq \mathbf{n} \), too.

Conversely, defining \( \Pi_k \) with (4.18), by using (4.1) and the orthogonality conditions (3.1), it is easy to see that

\[
[\Pi_n, x^{k_1}]_v = \int_{\Gamma} \Pi_n(x) x^{k_1} (iz)^{-1} w_v(z) \, dz = i \int_{A} \Pi_n(x) x^{k_1-1} w_v(x) \, dx = i (P_n(x), x^{k_1-1})_v = 0, \quad 0 \leq k_i < n_i,
\]

for \( v = 1, 2, \ldots, r. \)

Let \( \mathbf{n} \) be a multi–index, and let \( W \) be an admissible system for \( \mathbf{n} \). If the type II multiple orthogonal polynomial on the semicircle \( \Pi_n \) with respect to \( W \) is uniquely determined, such system \( W \) is an appropriate system for \( \mathbf{n} \). In what follows we consider only appropriate systems. The uniqueness of the corresponding type I multiple orthogonal polynomials on the semicircle for an appropriate system follows from Lemma 4.1.

We finish this section proving certain biorthogonality between the type II multiple orthogonal polynomials on the semicircle and the type I functions given by (4.9).

**Theorem 4.2.** Suppose that \( \mathbf{n} \) and \( \mathbf{m} \) are two multi–indices and that \( W = \{w_1, w_2, \ldots, w_r\} \) is an appropriate system for the both multi–indices \( \mathbf{n} \) and \( \mathbf{m} \). Then the following biorthogonality holds

\[
\int_{\Gamma} \Pi_n(z) B_m(z) (iz)^{-1} \, dz = \begin{cases} 
0, & \text{if } \mathbf{m} \preceq \mathbf{n}, \\
0, & \text{if } |\mathbf{n}| \leq |\mathbf{m}| - 2, \\
1, & \text{if } |\mathbf{n}| = |\mathbf{m}| - 1,
\end{cases}
\]

(4.19)

where \( \Pi_n(z) \) is the corresponding type II multiple orthogonal polynomial on the semicircle, and \( B_m(z) \) the corresponding type I function, given by (4.9).

**Proof.** By using (4.9) we have

\[
\int_{\Gamma} \Pi_n(z) B_m(z) (iz)^{-1} \, dz = \sum_{v=1}^{r} \int_{\Gamma} \Pi_n(z) B_m(z) (iz)^{-1} w_v(z) \, dz.
\]

Since every \( B_{m,v}(z) \) has degree \( m_v - 1 \), whenever \( \mathbf{m} \preceq \mathbf{n} \) we have

\[
\int_{\Gamma} \Pi_n(z) B_{m,v}(z) (iz)^{-1} w_v(z) \, dz = 0, \quad v = 1, 2, \ldots, r,
\]

due to the fact that \( m_v - 1 \leq n_v - 1, \quad v = 1, 2, \ldots, r \), and the type II orthogonality conditions (4.14), which proves the first result.

The degree of type II multiple orthogonal polynomial on the semicircle \( \Pi_n(z) \) is \(|\mathbf{n}|\), for \(|\mathbf{n}| \leq |\mathbf{m}| - 2 \) from type I orthogonality condition (4.10) we get
\[ \int \Pi_n(z) B_m(z) (iz)^{-1} \, dz = 0. \]

If \(|n| = |m| - 1\), then
\[ \int \Pi_n(z) B_m(z) (iz)^{-1} \, dz = \int x^{m-1} B_m(z) (iz)^{-1} \, dz = 1, \]
where the first equality follows from orthogonality (4.10) and the last equality follows form (4.11).

\[ \square \]

5. Recurrence relations

In this section we give recurrence relations for multiple orthogonal polynomials on the semicircle. Recurrence relations for the type II multiple orthogonal polynomials on the semicircle with respect to nearly diagonal multi–indices have already been known (see [15,16]). Here, we repeat that recurrence relations for completeness, and for arbitrary multi–indices we prove so called nearest neighbor recurrence relations for the both type I and type II multiple orthogonal polynomials on the semicircle.

5.1. Recurrence relations in the case of nearly diagonal multi–indices

Let \( m \) be a nonnegative integer and let \( W \) be an appropriate system for all nearly diagonal multi–indices \( \mathbf{d}(m) \). By \( \mathbf{x} \) we denote the nearly diagonal multi–index corresponding to \( m \). By \( \Pi_m \) we denote the corresponding type II multiple orthogonal polynomial on the semicircle \( \Pi_m(z) \) with respect to \( W \).

Taking \( [f, g]_{m, \ell} = [f, g]_{\ell} \) for each \( \ell \in \mathbb{Z} \), the following theorem holds.

**Theorem 5.1.** Let \( m \) be a nonnegative integer and let \( W \) be an appropriate system for all nearly diagonal multi–indices \( \mathbf{d}(k) \leq \mathbf{d}(m+1) \). The type II multiple orthogonal polynomials on the semicircle \( \{\Pi_k\} \) with nearly diagonal multi–indices satisfy the recurrence relation

\[ \Pi_{m+1}(z) = (z - x_{m,\ell}) \Pi_m(z) - \sum_{k=0}^{r-1} \alpha_{m,k} \Pi_{m-r+k}(z), \quad m \geq 0, \quad (5.1) \]

with the initial conditions \( \Pi_0(z) = 1 \), \( \Pi_{-1}(z) = \Pi_{-2}(z) = \cdots = \Pi_{-r}(z) = 0 \). Recurrence coefficients are given as follows

\[ \alpha_{m,0} = \frac{z \Pi_m, \Pi_{(m-r)/r}}{\Pi_{m-r}, \Pi_{(m-r)/r}} \]

and

\[ \alpha_{m,k} = \frac{z \Pi_m - \sum_{j=0}^{k-1} \alpha_{m,j} \Pi_{m-r+j}, \Pi_{(m-r+j)/r}}{\Pi_{m-r+k}, \Pi_{(m-r+j)/r}} \]

for \( j = m - \ell r \in \{0, 1, \ldots, r - 1\} \).

For calculating recurrence coefficients it is necessary to calculate inner products (5.2) and (5.3), i.e., to calculate the integrals of the following type \( \int \Pi_k(z)(iz)^{-1} w_r(z) \, dz \). For \( l \geq 1 \), because of (4.1), these integrals could be calculated exactly, except for rounding errors, by using the corresponding Gauss–Christoffel quadrature rules. For \( l = 0 \) one has

\[ \int \Pi_k(z)(iz)^{-1} w_r(z) \, dz = \mu_0^{(y)} \Pi_k(0) + i \int_1^{1} \frac{\Pi_k(x) - \Pi_k(0)}{x} w_r(x) \, dx. \]

Moments \( \mu_0^{(y)} \) could be calculated by using (4.6). For the calculation of the integrals \( \int_{1}^{1} (\Pi_k(x) - \Pi_k(0))/x w_r(x) \, dx \) Gauss–Christoffel quadrature rules could be used, too.

Putting \( m = 0, 1, \ldots, n - 1 \) in (5.1) we obtain

\[ H_n^C = \begin{bmatrix} \Pi_0(z) \\ \Pi_1(z) \\ \vdots \\ \Pi_{n-1}(z) \end{bmatrix} \begin{bmatrix} \Pi_0(z) \\ \Pi_1(z) \\ \vdots \\ \Pi_{n-1}(z) \end{bmatrix} = \begin{bmatrix} 0 \\ \Pi_0(z) \\ \vdots \\ \Pi_{n-1}(z) \end{bmatrix}, \]

i.e.,

\[ H_n^C \Pi_n(z) = z \Pi_n(z) - \Pi_n(z) e_n, \]

where \( \Pi_n(z) = [\Pi_0(z) \, \Pi_1(z) \, \cdots \, \Pi_{n-1}(z)]^T \), and \( H_n^C \) is the following (lower) banded complex Hessenberg matrix of order \( n \):
\[
H^C_n = \begin{bmatrix}
\alpha_{0,r} & 1 \\
\alpha_{1,r-1} & \alpha_{1,r} & 1 \\
\vdots & \ddots & \ddots & \ddots \\
\alpha_{r,0} & \cdots & \alpha_{r,r-1} & \alpha_{r,r} & 1 \\
\alpha_{r+1,0} & \cdots & \alpha_{r+1,r-1} & \alpha_{r+1,r} & 1 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
\alpha_{n-2,0} & \cdots & \alpha_{n-2,r-1} & \alpha_{n-2,r} & 1 \\
\alpha_{n-1,0} & \cdots & \alpha_{n-1,r-1} & \alpha_{n-1,r} & 1 \\
\end{bmatrix}
\]

Let \( \psi^{(n)}_{sk} \), \( k = 1, \ldots, n \), be the zeros of \( \Pi_n(z) \). Then (5.4) reduces to the following eigenvalue problem:

\[
\psi^{(n)}_{sk} \mathbf{H}_n \left( \psi^{(n)}_{sk} \right) = H^C_n \mathbf{P}_n \left( \psi^{(n)}_{sk} \right).
\]

Therefore, \( \psi^{(n)}_{sk} \) are eigenvalues of the matrix \( H^C_n \) and \( \mathbf{P}_n \left( \psi^{(n)}_{sk} \right) \) are the corresponding eigenvectors. It is easy to obtain the determinant representation \( \Pi_n(z) = \det(zI_n - H^C_n) \), where \( I_n \) is the identity matrix of the order \( n \).

**Remark 5.1.** Let us notice that in the case of nearly diagonal multi–indices all recurrence coefficients are represented in terms of type II multiple orthogonal polynomials on the semicircle. This fact provides us with a simple method for constructing of type II multiple orthogonal polynomials by using the discretized Stieltjes–Gautschi procedure (cf. [14,5]) as it was described. This is not the case with nearest neighbor recurrence relations for general multi–indices, where for the computation of the corresponding recurrence coefficients the both type I and type II multiple orthogonal polynomials on the semicircle are necessary.

**5.2. Nearest neighbor recurrence relations**

Now, we prove the recurrence relations for multiple orthogonal polynomials on the semicircle, analogous to those proved in [9] for multiple orthogonal (real) polynomials.

**Theorem 5.2.** Let \( \mathbf{n} \) be a multi–index. Choose \( \lambda \in \{1, 2, \ldots, r\} \) and suppose that \( W \) is an appropriate system for all multi–indices \( \mathbf{m} \leq \mathbf{n} + \mathbf{e}_\lambda \). Let \( \sigma_j, j = 1, 2, \ldots, r \), be multi–indices given by (3.2), where if \( \mathbf{n} \) has entries equal to 0, permutation of indices for \( \sigma_r \) is chosen according to Remark 3.1. Then the type II multiple orthogonal polynomials on the semicircle satisfy the recurrence relation

\[
\Pi_{\mathbf{n+e}_\lambda}(z) = \left( z - \alpha^{(n)}_{\mathbf{m},0} \right) \Pi_{\mathbf{n}}(z) - \sum_{j=1}^{r} \alpha_{\mathbf{n},\sigma_j} \Pi_{\mathbf{n}-\sigma_j}(z),
\]

(5.5)

with the initial conditions \( \Pi_0(z) = 1 \), \( \Pi_{-\sigma_j}(z) = 0 \), \( j = 1, 2, \ldots, r \). The recurrence coefficients are given by

\[
\alpha^{(n)}_{\mathbf{n},0} = \int_{\Gamma} \mathfrak{g}_n(z) \mathbf{B}_{\mathbf{n},\mathbf{e}_\lambda}(z)(iz)^{-1} \, dz,
\]

(5.6)

\[
\alpha_{\mathbf{n},j} = \int_{\Gamma} \mathfrak{g}_n(z) \mathbf{B}_{\mathbf{n}-\sigma_j}(z)(iz)^{-1} \, dz, \quad j = 1, 2, \ldots, r,
\]

(5.7)

where \( \mathbf{B}_n(z) \) is the corresponding type I function (4.9) and \( \sigma_0 = \mathbf{0} \).

**Proof.** Let \( \mathbf{m}_k, k = 0, 1, \ldots, |\mathbf{n}| \), be multi–indices, the same as in proof of Theorem 4.1, i.e., \( \mathbf{m}_0 = \mathbf{0} = (0, 0, \ldots, 0) \); \( |\mathbf{m}_k| = k \); \( \mathbf{m}_{k+1} = \mathbf{m}_k + \mathbf{e}_\lambda \), for some \( \lambda \); \( \mathbf{m}_k \leq \mathbf{n} \) for all \( k \); and the last \( r + 1 \) multi–indices are given by \( \mathbf{m}_n = \mathbf{n}, \mathbf{m}_{n+\sigma_j} = \mathbf{n} - \sigma_j \), \( \sigma_j = 1, 2, \ldots, r \). The type II multiple orthogonal polynomials on the semicircle \( \Pi_{\mathbf{m}_k}, k = 0, 1, \ldots, |\mathbf{n}| \), form a basis of the linear space \( \mathbb{F}_n \). Since we work with monic polynomials, \( z \Pi_n(z) - \Pi_{\mathbf{n}+\mathbf{e}_\lambda}(z) \) is a polynomial of degree at most \( |\mathbf{n}| \), so

\[
z \Pi_n(z) - \Pi_{\mathbf{n}+\mathbf{e}_\lambda}(z) = \sum_{k=0}^{|\mathbf{n}|} c_{n,k} \Pi_{\mathbf{m}_k}(z)
\]

(5.8)

for some coefficients \( c_{n,k}, k = 0, 1, \ldots, |\mathbf{n}| \). Since \( \mathbf{m}_j \leq \mathbf{m}_k \) whenever \( j \leq k \), and \( |\mathbf{m}_j| < |\mathbf{m}_k| \) if and only if \( j < k \), according to Theorem 4.2 we have

\[
\int_{\Gamma} \Pi_{\mathbf{m}_j}(z) \mathbf{B}_{\mathbf{m}_j}(z)(iz)^{-1} \, dz = 0, \quad j < k.
\]
\[
\int_{\Gamma} \Pi_m(z) B_m(z) (iz)^{-1} \, dz = 0, \quad k \leq j - 2,
\]
\[
\int_{\Gamma} \Pi_m(z) B_m(z) (iz)^{-1} \, dz = 1, \quad k = j - 1,
\]
which means that after multiplying the both hand sides of (5.8) by \( B_m(z)(iz)^{-1}, \ j = 1, 2, \ldots, |n|, \) and integrating over \( \Gamma \) we have
\[
\int_{\Gamma} z \Pi_n(z) B_m(z) (iz)^{-1} \, dz = c_{n-j}, \quad j = 1, 2, \ldots, |n|,
\]  
(5.9)
i.e.,
\[
\sum_{\nu=1}^{r} \int_{\Gamma} \Pi_n(z) z B_{m_{\nu}}(z)(iz)^{-1} w_\nu(z) \, dz = c_{n-j}, \quad j = 1, 2, \ldots, |n|.
\]
For \( j \leq |n| - r \) we have \( m_{\nu} \geq m_{n-r} = (n_1 - 1, n_2 - 1, \ldots, n_r - 1), \) hence \( z B_{m_{\nu}}(z) \) is a polynomial of degree less than or equal to \( n_r - 1 \) and
\[
\sum_{\nu=1}^{r} \int_{\Gamma} \Pi_n(z) z B_{m_{\nu}}(z)(iz)^{-1} w_\nu(z) \, dz = 0, \quad j \leq |n| - r,
\]
because of the orthogonality conditions (4.14). This means that \( c_{n-j} = 0 \) for all \( j \leq |n| - r, \) and (5.8) reduces to
\[
\Pi_{n,e}(z) = z \Pi_n(z) - \sum_{k=-|n|}^{\nu} c_{n-k} \Pi_m(z).
\]
Finally, denoting \( z_{n-j} = c_{n-m_{n-j}}: j = 1, 2, \ldots, r, \) and \( z_{n-0}^{(2)} = c_{n,m}, \) we get recurrence relation (5.5).

From (5.9) we get directly (5.7). Multiplying (5.8) by \( B_{n,e}(z)(iz)^{-1} \) and integrating over \( \Gamma \) we get (5.6) because of Theorem 4.2. □

**Remark 5.2.** It is obvious from (5.7) that recurrence coefficients \( z_{n-j}: j = 1, 2, \ldots, r, \) do not depend on \( \lambda. \) Therefore,
\[
\Pi_{n,e_1}(z) - \Pi_{n,e_2}(z) = \delta_{j_1,j_2} \left( z_{n-0}^{(2)} - z_{n-0}^{(1)} \right) \Pi_n(z),
\]
where \( \delta_{ij} \) is Kronecker’s delta.

It is known that recurrence relation of order \( r + 1 \) always has \( r + 1 \) linearly independent solutions. Thus, besides the type II multiple orthogonal polynomials on the semicircle, recurrence relation (5.5) has \( r \) other solutions. The functions
\[
\Phi_{n}^{(v)}(z) = \int_{\Gamma} \Pi_n(\zeta) w_{v}(\zeta) d\zeta, \quad v = 1, 2, \ldots, r,
\]
are called the functions of the second kind, and it can be easily shown that they satisfy the same recurrence relation (5.5) (but with different initial conditions). Indeed,
\[
z \Phi_{n}^{(v)}(z) = \int_{\Gamma} \Pi_n(\zeta)(iz)^{-1} w_{v}(\zeta) d\zeta + \int_{\Gamma} \frac{z \Pi_n(\zeta)}{Z - \zeta} w_{v}(\zeta) d\zeta, \quad v = 1, 2, \ldots, r.
\]
The first integral on the right hand side of the previous equation is equal to zero whenever \( n_r > 0 \) and by using (5.5) we get
\[
z \Phi_{n}^{(v)}(z) = \Phi_{n,e}(z) + z_{n-0}^{(2)} \Phi_{n}(z) + \sum_{j=1}^{r} z_{n,j} \Phi_{n,j}(z),
\]
i.e., \( \Phi_{n}^{(v)}, \quad v = 1, 2, \ldots, r, \) satisfy recurrence relation (5.5) whenever \( n_r > 0.\)

**Theorem 5.3.** Let \( n \neq 0 \) be a multi-index. Choose \( \lambda \in \{1, 2, \ldots, r\} \) such that \( n_\lambda 
eq 0, \) and suppose that \( W \) is an appropriate system for all multi-indices \( m \leq n + (1, 1, \ldots, 1). \) Let \( \sigma_j, \ j = 1, 2, \ldots, r, \) be multi-indices given by (3.2). Then the type I functions \( \{B_m\}, \) given by (4.9), satisfy the recurrence relation
\[
B_{n,e_1}(z) = \left( z - \beta_{n-0}^{(2)} \right) B_n(z) - \sum_{j=1}^{r} \beta_{n,j} B_{n,\sigma_j}(z),
\]
(5.10)
with \( B_0(z) = 0 \) and \( B_n(z), \ j = 1, 2, \ldots, r, \) given by (4.13). The recurrence coefficients are given by
\[
\beta_{n-0}^{(2)} = \int_{\Gamma} z B_n(z) \Pi_{n-e_1}(z)(iz)^{-1} dz,
\]
(5.11)
\[
\beta_{nj} = \int_{\Gamma} zB_n(z)\Pi_{n,j-}(z)(iz)^{-1}\,dz, \quad j = 1, 2, \ldots, r, \tag{5.12}
\]
where \(\Pi_{m}(z)\) are the corresponding type II multiple orthogonal polynomial on the semicircle and \(\sigma_0 = 0\).

**Proof.** Let \(m_k, k = 0, 1, \ldots, |n| + r\), be multi–indices such that \(m_0 = 0; |m_k| = k; m_k \leq m_{k+1}; m_n = n; m_{n-1} = n - e_v\). Then,

\[
zB_n(z) = \sum_{k=1}^{|n|+r} d_{n,k}B_{m_k}(z).
\]

Multiplying the both hand sides of the previous equation by \(\Pi_{m}(z)(iz)^{-1}\) and integrating over \(\Gamma\), because of Theorem 4.2, we get

\[
d_{n,j-1} = \int_{\Gamma} B_n(z)z\Pi_{m}(z)(iz)^{-1}\,dz, \quad j = 1, 2, \ldots, |n| + r.
\]

These integrals are equal to 0 whenever \(j + 1 \leq |n| - 2\), due to (4.10), hence,

\[
zB_n(z) = \sum_{j=|n|-1}^{|n|+r} d_{n,j}B_{m_j}(z).
\]

It is easy to see because of normalization (4.11) that \(d_{n,|n|-1} = 1\). Denoting \(\beta_{nj} = d_{n,|n|+j}, j = 1, 2, \ldots, r\), and \(\beta_{n0} = d_{n,0}\), we get (5.10).

Formulae (5.11) and (5.12) follow easily. \(\square\)

**Remark 5.3.** As well as in the case of recurrence coefficients for the type II multiple orthogonal polynomials on the semicircle, only one coefficient \(\beta_{n0}\) depends on \(\lambda\), which implies

\[
B_{n-e_v}(z) - B_{n-e_v}(z) = \delta_{n-j,1} \left( \beta_{n0}^{(j)} - \beta_{n0}^{(j+1)} \right) B_n(z),
\]

where \(\delta_{n,j}\) is Kronecker’s delta.

Similarly, the following recurrence relation for the type I functions (4.9) can be proved, where the type I functions with multi–indices \(e_v\), \(v = 1, 2, \ldots, r\), are used instead of \(\sigma_v\), \(v = 1, 2, \ldots, r\).

**Theorem 5.4.** Let \(n \neq 0\) be a multi–index. Choose \(i \in \{1, 2, \ldots, r\}\) such that \(n_i \neq 0\), and suppose that \(W\) is an appropriate system for all multi–indices \(m \leq n\) and for \(n + e_v, v = 1, 2, \ldots, r\). Then the type I functions \((B_m)\), given by (4.9), satisfy the recurrence relation

\[
B_{n-e_v}(z) = \left( z - \delta_{n,0}^{(j)} \right) B_n(z) - \sum_{j=1}^r \delta_{nj}B_{n-e_j}(z), \tag{5.13}
\]

with \(B_0(z) = 0\) and \(B_{e_r}(z)\), \(j = 1, 2, \ldots, r\), given by (4.12). The recurrence coefficients are given by

\[
\delta_{n0}^{(j)} = \int_{\Gamma} zB_n(z)\Pi_{n-e_j}(z)(iz)^{-1}\,dz,
\]

\[
\delta_{nj} = \frac{b_{nj}}{b_{n-e_j}}, \quad j = 1, 2, \ldots, r,
\]

where \(\Pi_{m}(z)\) is the corresponding type II multiple orthogonal polynomial on the semicircle, and \(b_{m,j}\) is the leading coefficient of the polynomial \(B_m(z)\) (of degree \(m_j - 1\)).

**Proof.** Let \(m_k, k = 0, 1, \ldots, |n|\), be multi–indices such that \(m_0 = 0; |m_k| = k; m_k \leq m_{k+1}; m_n = n; m_{n-1} = n - e_v\). Expanding

\[
zB_n(z) = \sum_{k=1}^{|n|} d_{n,k}B_{m_k}(z) + \sum_{j=1}^r d_{nj}B_{n-e_j}(z),
\]

by using biorthogonality (4.19), orthogonality (4.10), and the fact that \(zB_{n,v}\) and \(B_{n-e_j,v}\) are polynomials of degree \(n_j\), it is easy to get what is stated. \(\square\)
Finally, we finish this section proving a similar recurrence relation for the type II multiple orthogonal polynomials on the semicircle.

**Theorem 5.5.** Let \( \mathbf{n} \) be a multi-index. Choose \( i \in \{1, 2, \ldots, r\} \) and suppose that \( W \) is an appropriate system for all multi-indices \( \mathbf{m} \leq \mathbf{n} + \mathbf{e} \). Then the type II multiple orthogonal polynomials on the semicircle satisfy the recurrence relation

\[
\Pi_{\mathbf{n}+\mathbf{e}}(z) = (z - \gamma_{\mathbf{n}0}^{(i)}) \Pi_{\mathbf{n}}(z) - \sum_{j=1}^{r} \gamma_{\mathbf{n}j} \Pi_{\mathbf{n}-\mathbf{e}}(z),
\]

with the initial conditions \( \Pi_0(z) = 1 \), and \( \Pi_{-\mathbf{e}}(z) = 0 \), \( v = 1, 2, \ldots, r \). The recurrence coefficients are given by

\[
\gamma_{\mathbf{n}0}^{(i)} = \int_{\Gamma} z \Pi_{\mathbf{n}}(z) B_{\mathbf{n}+\mathbf{e}}(iz)^{-1} \, dz.
\]

\[
\gamma_{\mathbf{n}j} = \left[ \frac{\Pi_{\mathbf{n}j}}{\Pi_{\mathbf{n}-\mathbf{e}j, \mathbf{n}0-1}} \right]_{j}, \quad j = 1, 2, \ldots, r,
\]

where \( B_{\mathbf{m}}(z) \) is the corresponding type I function given by (4.9).

**Proof.** Since \( \Pi_{\mathbf{n}}(z) \) and \( \Pi_{\mathbf{n}+\mathbf{e}}(z) \) are monic polynomials and all multi-indices \( \mathbf{m} \leq \mathbf{n} + \mathbf{e} \) are normal, then \( z \Pi_{\mathbf{n}}(z) - \Pi_{\mathbf{n}+\mathbf{e}}(z) \) is a polynomial of degree at most \( |\mathbf{n}| \). Let \( \gamma_{\mathbf{n}0}^{(i)} \) be such that \( \Pi_{\mathbf{n}0}(z) = z \Pi_{\mathbf{n}0}(z) - \Pi_{\mathbf{n}+\mathbf{e}0}(z) = \gamma_{\mathbf{n}0}^{(i)} \Pi_{\mathbf{n}}(z) \) is a polynomial of degree less than or equal to \( |\mathbf{n}| - 1 \). Due to orthogonality conditions (4.14), it is easy to see that \( \left[ \Pi_{\mathbf{n}j}, z^k \right] \), \( k = 0, 1, \ldots, n_v - 2, \) for all \( v = 1, 2, \ldots, r \). Let us denote by \( \mathcal{P}_{|\mathbf{n}|} \) the linear space of all complex polynomials of degree at most \( |\mathbf{n}| - 1 \) which are orthogonal to all polynomials of degree less than or equal to \( n_v - 2 \) with respect to inner product \( \langle \cdot, \cdot \rangle_v \), \( v = 1, 2, \ldots, r \), which corresponds to the linear space of coefficients \( \tilde{a} = (a_0, a_1, \ldots, a_{n_v-1}) \in \mathbb{C}^{|\mathbf{n}|} \) of polynomials from \( \mathcal{P}_{|\mathbf{n}|} \), satisfying the homogeneous system of linear equations \( M_{\mathbf{n}} \tilde{a} = \tilde{0} \), where \( M_{\mathbf{n}} \) is obtained deleting \( r \) rows in the corresponding matrix \( M_{|\mathbf{n}|}^{(r)} \), given by (4.15) (deleting the last row in each \( M_{|\mathbf{n}|}^{(r)} \), \( v = 1, 2, \ldots, r \)). The rank of \( M_{\mathbf{n}} \) is \( |\mathbf{n}| - r \), which gives that the dimension of \( \mathcal{P}_{|\mathbf{n}|} \) is \( r \). It is easy to see that \( \Pi_{\mathbf{n}+\mathbf{e}j} \in \mathcal{P}_{|\mathbf{n}|} \), \( v = 1, 2, \ldots, r \), are linearly independent due to normality of \( \mathbf{n} \) (which implies that \( \Pi_{\mathbf{n}+\mathbf{e}j, \mathbf{n}0-1} \neq 0 \), \( v = 1, 2, \ldots, r \)). Therefore,

\[
z \Pi_{\mathbf{n}}(z) - \Pi_{\mathbf{n}+\mathbf{e}}(z) = \gamma_{\mathbf{n}0}^{(i)} \Pi_{\mathbf{n}0}(z) \sum_{j=1}^{r} \gamma_{\mathbf{n}j} \Pi_{\mathbf{n}-\mathbf{e}j},
\]

which gives (5.14). Multiplying the both hand sides of (5.14) by \( B_{\mathbf{n}+\mathbf{e}}(z)(iz)^{-1} \) and integrating over \( \Gamma \) we get (5.15), due to biorthogonality (4.19). Similarly, multiplying the both hand sides of (5.14) by \( z^{n_v-1}(iz)^{-1} W_v(z) \), \( k = 1, 2, \ldots, r \), and integrating over \( \Gamma \) we get (5.16), due to orthogonality conditions (4.14). \( \square \)

6. Applications

Motivated by paper of Borges [3] (see also [14,16]), we define and characterize an optimal set of quadrature rules on the semicircle. We consider the problem of numerically evaluating a set of \( r \) integrals over the semicircle with respect to distinct weight functions, but related to a common integrand:

\[
\int_{0}^{\pi} f(e^{i\theta}) w_v(e^{i\theta}) \, d\theta = \int_{\Gamma} f(z)(iz)^{-1} w_v(z) \, dz = \sum_{k=1}^{|\mathbf{n}|} \gamma_{\mathbf{n}k} f(\zeta_{v,k}) + R_{|\mathbf{n}|} f,
\]

for \( v = 1, 2, \ldots, r \). Considering a performance ratio (see [3]), defined as

\[
R = \frac{\text{Overall degree of precision} + 1}{\text{Number of integrandevaluations}},
\]

it is easy to see that in the case of \( r \) Gaussian quadrature rules (i.e., quadrature rules with maximal degree of exactness \( R = 2/r \)) and, hence, \( R < 1 \) for all \( r > 2 \). Selecting a set of \( |\mathbf{n}| \) distinct nodes, common for all quadrature rules, one can increase values of \( R \).

**Definition 6.1.** Let \( \mathbf{n} \) be a multi-index and let \( W \) be an appropriate system for all multi-indices \( \mathbf{m} \leq \mathbf{n} \). The following set of quadrature rules

\[
\int_{0}^{\pi} f(e^{i\theta}) w_v(e^{i\theta}) \, d\theta = \int_{\Gamma} f(z)(iz)^{-1} w_v(z) \, dz = \sum_{k=1}^{|\mathbf{n}|} \gamma_{\mathbf{n}k} f(\zeta_{v,k}) + R_{|\mathbf{n}|} f,
\]

(6.1)
Theorem 6.1. Let \( n \) be a multi–index and let \( W \) be an appropriate system for \( n \). A set of quadrature rules (6.1) is the optimal set on the semicircle with respect to \( (W, n) \) if and only if:

1° all rules are exact for all polynomials from \( P_{m-1} \);
2° \( \Pi_{n}(z) = \prod_{k=1}^{m}(z - \xi_k) \) is the type II multiple orthogonal polynomial on the semicircle with respect to \( W \).

Proof. First we suppose that quadrature rules (6.1) form the optimal set on the semicircle with respect to \( (W, n) \). Then, for each \( v = 1, 2, \ldots, r \), the corresponding quadrature rule with respect to the weight function \( w_v \) is exact for all polynomials of degree less than or equal to \( |n| + n_v - 1 \), hence, it is exact for those of degree less than or equal to \( |n| - 1 \). Thus, 1° is proved.

For each \( v = 1, 2, \ldots, r \) assume that \( \rho(z) \) is a polynomial of degree less than or equal to \( n_v - 1 \). Then the polynomial \( \Pi_{n}(z) \rho(z) \) is a polynomial of degree less than or equal to \( |n| + n_v - 1 = |n|-1 \). Since the corresponding quadrature rule, with respect to the weight function \( w_v \), \( v = 1, 2, \ldots, r \), is exact for all such polynomials and \( \Pi_{n}(z) = 0 \), \( k = 1, 2, \ldots, |n| \), it follows that

\[
\int_{\Gamma} \Pi_{n}(z)\rho(z)(iz)^{-1}w_v(z)\,dz = \sum_{k=1}^{m} \sigma_{v,k}\Pi_{n}(z)\rho(z) = 0, \quad v = 1, 2, \ldots, r,
\]

which gives 2°.

Let us now suppose that 1° and 2° hold for (6.1).

Let \( \Psi_v(z) \in P_{m-1} \) for \( v = 1, 2, \ldots, r \). The polynomial \( \Psi_v(z) \) can be represented in the form

\[
\Psi_v(z) = \Pi_{n}(z)q_v(z) + r_v(z),
\]

where \( q_v \in P_{m-1} \) and \( r_v \in P_{m-1} \).

Now we have that

\[
\int_{\Gamma} \Psi_v(z)(iz)^{-1}w_v(z)\,dz = \int_{\Gamma} \left[ \Pi_{n}(z)q_v(z) + r_v(z) \right](iz)^{-1}w_v(z)\,dz = \int_{\Gamma} \Pi_{n}(z)q_v(z)(iz)^{-1}w_v(z)\,dz + \int_{\Gamma} r_v(z)(iz)^{-1}w_v(z)\,dz,
\]

for all \( v = 1, 2, \ldots, r \). From 2° it follows that

\[
\int_{\Gamma} \Pi_{n}(z)q_v(z)(iz)^{-1}w_v(z)\,dz = 0, \quad v = 1, 2, \ldots, r,
\]

and, since \( r_v \in P_{m-1} \), from 1° we have that

\[
\int_{\Gamma} r_v(z)(iz)^{-1}w_v(z)\,dz = \sum_{k=1}^{m} \sigma_{v,k}r_v(\xi_k), \quad v = 1, 2, \ldots, r,
\]

and, hence,

\[
\int_{\Gamma} \Psi_v(z)(iz)^{-1}w_v(z)\,dz = \sum_{k=1}^{m} \sigma_{v,k}r_v(\xi_k), \quad v = 1, 2, \ldots, r.
\]
Finally, since $\Pi_n(z_k) = 0$, $k = 1, 2, \ldots, |n|$, it follows $\Psi_n(z_k) = r_v(z_k)$, $k = 1, 2, \ldots, |n|$, and we obtain

$$\int_{\Gamma} \Psi_v(z) \overline{\Psi_w(z)} \, dz = \sum_{k=1}^{n} \sigma_{v,k} \Psi_v(z_k), \quad v = 1, 2, \ldots, r,$$

i.e., quadrature rule for the weight $w_v$ is exact for all polynomials of degree less than or equal to $|n| + n_v - 1$, $v = 1, 2, \ldots, r$. Therefore, (6.1) is the optimal set with respect to $(W, n)$. □

We see from Theorem 6.1 that nodes of the optimal set of quadrature rules with respect to $(W, n)$ are the zeros of the corresponding type II multiple orthogonal polynomial on the semicircle $\Pi_n$. Numerical experiments with systems of appropriate sets of Jacobi weight functions indicate that the zeros of the corresponding type II multiple orthogonal polynomial are simple and lie in $D$. Here we give an example of optimal set of $r = 2$ quadrature rules on the semicircle with respect to Jacobi weights $w_\beta(z) = (1 - z^2)^{\beta_1} (1 + z)^{\beta_2}$, $v = 1, 2$, where $z = 1$, $\beta_1 = 0$, $\beta_2 = 1/2$ for $n = (5, 4)$. Since we have nearly diagonal multi–index, nodes $z_k$, $k = 1, 2, \ldots, 9$, can be calculated as eigenvalues of the corresponding banded complex Hessenberg matrix, as it was explained in Subsection 5.1. When the nodes are known, according to Theorem 6.1, weight coefficients can be obtained by solving the following Vandermonde systems of linear equations (here $|n| = 5 + 4 = 9$):

$$V(z_1, z_2, \ldots, z_9) \begin{bmatrix} \sigma_{v,1} \\ \sigma_{v,2} \\ \vdots \\ \sigma_{v,9} \end{bmatrix} = \begin{bmatrix} \mu_0^{(v)} \\ \mu_1^{(v)} \\ \vdots \\ \mu_8^{(v)} \end{bmatrix}, \quad v = 1, 2,$$

where $\mu_k^{(v)}$, $k = 0, 1, \ldots, 8$, are moments given by (4.5). The parameters of the corresponding optimal set of quadrature rules are given in Table 1.

For the further research it is interesting to study in detail the zeros of type II multiple orthogonal polynomials and to consider numerical procedure for constructing introduced multiple orthogonal systems as well as for constructing considered quadrature rules for general multi–index.

References