Legendre–Galerkin method for the linear Fredholm integro-differential equations

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1. Introduction

The boundary value problems in terms of integro-differential equations have many practical applications such as mechanics, physics, chemistry, astronomy, biology, economics, potential theory, engineering problems [19] and electrostatics [11]. The existence and uniqueness of the solutions for these problems were discussed by Agarwal [1,2].

The present work is motivated by the desire to obtain numerical solutions to boundary value problems for second-order Fredholm integro-differential equations via Legendre–Galerkin method. We will consider the numerical solution of a class of linear Fredholm integro-differential boundary value problems in the form

\[ \sum_{i=0}^{n} \mu_i(x)u^{(i)}(x) = f(x) + \lambda \int_{-1}^{1} k(x,t)u(t)dt, \]

\[ \sigma u(-1) = \sigma(\sigma - 1)u(1) = 0, \]

where \( \mu_i(x), f(x), k(x,t) \) and \( u(x) \) are continuous functions in \( L^2 \) space, \( \sigma = 0, 1, 2 \) and \( \lambda \) is a parameter.

Many numerical methods have been developed for solving the integro-differential Eq. (1.1) with Fredholm type numerically. Each of these methods has its inherent advantages and disadvantages and the search for alternative, more general, easier and more accurate methods is a continuous and ongoing process. Next, we present a selective review of the most recent main methods. These methods include: Adomian decomposition [9], variational iteration method [5], homotopy analysis method [11], Chebyshev and Taylor collocation [22], Haar wavelet [15], sinc-collocation method [17], Tau method [10], Taylor’s series expansion [8,13], hybrid Taylor and Block Pulse functions [14] and integral mean value [3].

In recent years, a lot of attention has been devoted to the study of Legendre methods to investigate various scientific models. Using these methods made it possible to solve differential equations of Lane–Emden type [24], second and fourth order
equations [21], Cahn–Hilliard equations with Neumann boundary conditions [28], Fredholm integral [12], Helmholtz equation [4], second kind Volterra integral equations [29], high-order linear Fredholm integro-differential [23] and Abel’s integral equation [25].

This paper is organized as follows: in Section 2, we present the preliminaries of Legendre polynomials, new lemmas and theorems required for our subsequent development. In Section 3, convergence and error estimation of the method are described in detail. Section 4 is devoted to derivation of the discrete system. Section 5 presents appropriate techniques to treat nonhomogeneous boundary conditions for case \( n = 2 \). Section 6 shows the accuracy of the proposed method using numerical examples and making comparisons with other methods. Section 7 gives a brief conclusion. Note that we have computed the numerical results by Matlab programming.

2. Legendre function preliminaries

Orthogonal polynomials are widely used in applications in mathematics, mathematical physics, engineering and computer science. One of the most common orthogonal polynomial set is the Legendre polynomials. The Legendre polynomials \( P_n(x) \) satisfy the Legendre differential equation

\[
(1 - x^2)y'' - 2xy' + n(n - 1)y = 0, \quad -1 \leq x \leq 1, \quad n \geq 0
\]

with recurrence relations

\[
P_{n+1}(x) - P_{n-1}(x) = (2n + 1)P_n(x), \tag{2.1}
\]

\[
xP_n'(x) - P_{n-1}(x) = nP_n(x). \tag{2.2}
\]

These polynomial are orthogonal on \([-1, 1]\)

\[
\int_{-1}^{1} P_m(x)P_n(x)dx = \begin{cases} \frac{2}{2n+1}, & \text{if } m = n, \\ 0, & \text{if } m \neq n \end{cases} \tag{2.3}
\]

and

\[
\int_{-1}^{1} P_n(x)dx = \begin{cases} 2, & n = 0, \\ 0, & n > 0 \end{cases} \tag{2.4}
\]

Lemma 2.1. Let \( n, l \) and \( N \) be any two positive integer numbers such that \( n - l \leq N \) and \( l > 0 \), then

\[
\int_{-1}^{1} P_n(x)P_{n-1}(x)dx = 0.
\]

Proof. Integrating the left term by parts and using Eq. (2.4) can prove the above lemma. \( \square \)

Lemma 2.2. Let \( n, m \) and \( N \) be any two integer numbers such that \( n \geq m \) and \( n, m \leq N \), then

\[
\int_{-1}^{1} P_n(x)P_m'(x)dx = 0.
\]

Proof. First, let \( n = m \). Integrating the left hand side in Lemma 2.2 two times by parts yields

\[
\int_{-1}^{1} P_n(x)P_m'(x)dx = \frac{1}{2} [(P_n(x))^2]_{-1}^{1} = 0 \Rightarrow \int_{-1}^{1} P_n(x)P_m'(x)dx = 0.0
\]

Second, let \( n > m \). Using the recurrence Eq. (2.1) and (2.4), the left term in Lemma 2.2 can be written as follows:

\[
\int_{-1}^{1} P_n(x)P_m'(x)dx = \int_{-1}^{1} P_n(x)(2m + 1)P_{m-1}(x) + P_{m-2}(x))dx = \int_{-1}^{1} P_n(x)P_{m-2}(x)dx = \int_{-1}^{1} P_n(x)P_{m-4}(x)dx
\]

\[
= \cdots = \begin{cases} \int_{-1}^{1} P_n(x)P_0'(x)dx, & \text{if } m \text{ even}, \\ \int_{-1}^{1} P_n(x)P_1'(x)dx, & \text{if } m \text{ odd} \end{cases} = 0. \quad \square
\]
Theorem 2.1. Let \( n, m \) and \( N \) be any two integer numbers such that \( n, m \leq N \), then

\[
\begin{align*}
(1) \int_1^1 P'_n(x)P_m(x)dx &= \begin{cases} 2, & \text{if } n = m + i, \\ 0, & \text{if } n \neq m + i \text{ or } m \geq n \end{cases} \\
(2) \int_1^1 xP'_n(x)P_m(x)dx &= \begin{cases} \frac{2n}{x}, & \text{if } m = n, \\ 0, & \text{if } n = m + i \text{ or } m > n, \\ 2, & \text{if } n \neq m + i \end{cases} \\
(3) \int_1^1 xP'_n(x)P'_m(x)dx &= \begin{cases} 0, & \text{if } n = m \text{ or } m \neq n + i \text{ or } n \neq m + i \\ n(n + 1), & \text{if } m \neq n + i, m > n \\ m(m + 1), & \text{if } n \neq m + i, n > m \end{cases} \\
(4) \int_1^1 P'_n(x)P'_m(x)dx &= \begin{cases} 0, & \text{if } n = m \text{ or } m \neq n + i \text{ or } n \neq m + i \\ m(m + 1), & \text{if } n = m + i, m > n \\ n(n + 1), & \text{if } m = n + i, m > n \end{cases}
\end{align*}
\]

where \( i = 1, 3, 5, \ldots, 2k + 1 \leq N - m \)

Proof

(i) Integrating the left hand side for (i) by parts yields

\[
\int_1^1 P'_n(x)P_m(x)dx = [P_n(x)P_m(x)]_1^1 - \int_1^1 P_n(x)P'_m(x)dx = [1 + (-1)^{n-m+1}] - \int_1^1 P_n(x)P'_m(x)dx.
\]

For \( n = m + i, i = 1, 3, 5, \ldots \leq N - m \), by using Lemma (2.2), the integration (2.5) can be written as follows:

\[
\int_1^1 P'_n(x)P_m(x)dx = 2.
\]

As in the above case and Lemma (2.2) is considered but \( n = m + i, i = 0, 2, 4, \ldots \leq N - m \), yields

\[
\int_1^1 P'_n(x)P_m(x)dx = 0.
\]

For \( m \geq n \), previous cases must be considered besides Lemma 2.2. So, the right hand side of Eq. (2.5) is equal to zero.

(ii) Combine Lemma 2.2 and (i) besides recurrence Eq. (2.2). To prove (ii), the proof must be divide into four cases.

First, let \( m = n \), by taking back recurrence Eq. (2.2) and recalling the orthogonality of the Legendre polynomials, the first part of the above theorem can be drawn up as

\[
\int_1^1 xP'_n(x)P'_m(x)dx = m(n + 1), \quad \text{if } m \neq n + i, n > m
\]

In the next case, let \( n = m + i, i = 1, 3, 5, \ldots \leq N - m \), the proof is as that in the previous case but \( n = m + i \). The integration is equal to zero. On the other hand, Theorem 2.1(ii) can be proved in status \( m > n \) as in the previous two cases but the integration is zero. Finally, assume \( n = m + i, i = 2, 4, \ldots \leq N - m \), by getting back to recurrence Eq. (2.2), Lemma 2.2 and (i). Integrating by parts yields

\[
\int_1^1 xP'_n(x)P'_m(x)dx = [P_{n-1}(x)P_n(x)]_1^1 - \int_1^1 P_{n-1}(x)P'_n(x)dx = 2.
\]

(iii) The left term in the above lemma can be written by using recurrence Eq. (2.2) as

\[
\int_1^1 xP'_n(x)P'_m(x)dx = \int_1^1 nP_n(x)P'_n(x)dx + \int_1^1 P'_n(x)P'_m(x)dx
\]

for \( m = n \), recalling (i), Lemma 2.1 and integrating the second term in the right hand side for Eq. (2.6), the result is zero. In this case assume \( m > n \) and \( m = n + 1 \), as in the first case the result is equal to \( n(n + 1) \). In the next status let \( m > n \) and \( m \neq n + 1 \), the integration is equal to zero. Finally, at \( n > m \), the recurrence can be proved as in the above cases by interchanging \( n \) by \( m \) and \( m \) instead \( n \).

(iv) The integration in the left hand side in (iv) can be written by using the recurrence (2.2) as

\[
\int_1^1 P'_n(x)P'_m(x)dx = \int_1^1 xP'_{n+1}(x)P'_m(x)dx - (n + 1) \int_1^1 P_{n+1}(x)P'_m(x)dx
\]

then, by using (iii) and (i) into Eq. (2.7) and substituting each case of them, (iv) is proved. □
Theorem 2.2. Let \( n, m \) and \( N \) be any two integer numbers such that \( n, m \leq N \), then

\[
\begin{align*}
(i) \quad & \int_{-1}^{1} P_n'(x) P_m(x) \, dx = \begin{cases} n(n+1) - m(m+1), & \text{if } n \neq m + i, \\ 0, & \text{if } n = m + i \text{ or } m \geq n \end{cases} \\
(ii) \quad & \int_{-1}^{1} x P_n''(x) P_m(x) \, dx = \begin{cases} n(n+1) - m(m+1) - 2, & \text{if } n = m + i, \\ 0, & \text{if } n \neq m + i \text{ or } m \geq n \end{cases} \\
(iii) \quad & \int_{-1}^{1} x^2 P_n''(x) P_m(x) \, dx = \begin{cases} \frac{2n(n-1)}{n+1}, & \text{if } m = n, \\ n(n+1) - m(m+1) - 4, & \text{if } n = m + 1, \\ 0, & \text{if } n \neq m + 1 \text{ or } m > n. \end{cases}
\end{align*}
\]

where \( i = 1, 3, 5, \ldots, 2k + 1 \leq N - m - 1. \)

Proof

(i) The proof must be divided into four cases. First, let \( n = m + i, i = 2, 4, 6, \ldots, 2k + 1 \leq N - m \). Integrating the left hand side by parts twice and using Lemma 2.1 produce

\[
\int_{-1}^{1} P_n'(x) P_m(x) \, dx = [P_n'(x) P_m(x)]_{-1}^{1} - \int_{-1}^{1} P_n'(x) P_m'(x) \, dx = n(n+1) - [P_n'(x) P_m(x)]_{-1}^{1} + \int_{-1}^{1} P_n'(x) \, dx
\]

Second, let \( n = m + i, i = 1, 3, 5, \ldots, 2k + 1 \leq N - m \). As in the first case, but the value of \( i \) is odd numbers. So, the integration is zero.

Third, let \( m > n \). By using Eq. (2.4) and integrating the left side, the result is equal to zero.

Fourth, at \( m = n \), the value of the integration also is equal to zero by using the integration by parts.

(ii) To prove this point, Theorem 2.1 (i) and Theorem 2.1 (iii) are considered. Substitute each case in each of them after integrating the left side of the above theorem by parts.

(iii) Integrating the left term in the above theorem yields

\[
\int_{-1}^{1} x^2 P_n''(x) P_m(x) \, dx = \frac{n(n+1)}{2} [1 + (-1)^{m+n}] - 2 \int_{-1}^{1} xP_n'(x) P_m(x) \, dx - \int_{-1}^{1} x^2 P_n'(x) P_m'(x) \, dx. \tag{2.8}
\]

Moreover, by recalling recurrence Eq. (2.2) and using it in the last term in Eq. (2.8) then expanding the result yields

\[
\int_{-1}^{1} x^2 P_n'(x) P_m'(x) \, dx = nm \int_{-1}^{1} P_n(x) P_m(x) \, dx + n \int_{-1}^{1} P_n(x) P_m'(x) \, dx + m \int_{-1}^{1} P_m(x) P_n'(x) \, dx + \int_{-1}^{1} P_n'(x) P_m'(x) \, dx. \tag{2.9}
\]

by substituting Eq. (2.9) and Theorem 2.1(ii) into Eq. (2.8), then taking back the orthogonality of the Legendre polynomial (2.3) and using Theorem 2.1(i) and Theorem 2.1(iv), the result produces four cases. By collecting the four cases, the above theorem can be proved. \( \square \)

3. Convergence and error estimation

3.1. Convergence of Legendre polynomial

Consider Fredholm integral equation of the second kind by putting \( \sigma = 0 \) and \( \mu_0(x) = 1 \)

\[
u(x) = f(x) + \lambda \int_{-1}^{1} k(x, t) u(t) \, dt. \tag{3.1}\]

If we define

\[
\mathcal{N} : X \rightarrow X, \quad \mathcal{N}u = \lambda \int_{-1}^{1} k(x, t) u(t) \, dt,
\]

we can rewrite (3.1) as a form of

\[
(I - \lambda \mathcal{N}) u = f.
\]
**Theorem 3.1** [16]. Let $X$ be a normed space, $\mathcal{N} : X \rightarrow X$ a compact and $I - \lambda \mathcal{N}$ be injective. Then the inverse operator $(I - \lambda \mathcal{N})^{-1} : X \rightarrow X$ exist and is bounded.

By regard to the above theorem, we conclude the uniqueness of solution of Fredholm integral equation of the second kind.

**Theorem 3.2** [16]. Assume $\mathcal{N} : Y \rightarrow Y$ is bounded, with $Y$ a Banach space, and assume $\gamma - \mathcal{N} : Y \rightarrow Y$ is one to one and onto. Further assume
\[
||\mathcal{N} - P_n\mathcal{N}|| \rightarrow 0 \text{ as } n \rightarrow \infty,
\]
then for all sufficiently large $n$, say $n \geq N$, the operator $(\gamma - P_n\mathcal{N})^{-1}$ exists as a bounded operator from $Y$ to $Y$. Moreover, it is uniformly bounded:
\[
\sup_{n \geq N} ||(\gamma - P_n\mathcal{N})^{-1}|| < \infty.
\]

For the solution of $(\gamma - P_n\mathcal{N})x_n = P_ny$, $x_n \in Y$ and $(\gamma - \mathcal{N})x = y$;
\[
x - x_n = (\gamma - P_n\mathcal{N})^{-1}(x - P_n(x)),
\]
\[
\frac{|\gamma|}{||\gamma - P_n\mathcal{N}||} ||x - P_n(x)|| \leq ||x - x_n|| \leq |\gamma||(\gamma - P_n\mathcal{N})^{-1}|| ||x - P_n(x)||.
\]

**Lemma 3.1** ([6,16]). Let $x(t) \in H^k(-1, 1)$ (a Sobolev space) and let $x_n(t) = \sum_{i=0}^n a_i P_i(t)$ be the best approximation polynomial of $x(t)$ in the $L^2$-norm, then
\[
||x(t) - x_n(t)||_{L^2(-1, 1)} \leq c_0 h^{-k} ||x(t)||_{H^k(-1, 1)}
\]
where
\[
||x(t)||_{L^2(-1, 1)} = \left( \int_{-1}^1 x^2(t) dt \right)^{1/2},
\]
\[
||x(t)||_{H^k(-1, 1)} = \left( \sum_{i=0}^k \int_{-1}^1 |x^{(i)}(t)|^2 dt \right)^{1/2}.
\]
and $c_0$ is a positive constant, which depends on the selected norm and is independent of $x(t)$ and $n$.

By regard to the Lemma 3.1 we conclude that approximation rate of Legendre polynomials is $n^{-k}$, also Theorem 3.2 indicates that $||x - x_n||$ converge to zero at exactly the same speed as $||x - P_n(x)||$.

### 3.2. Error estimation of Legendre-Galerkin method

In this section an error estimator for the Legendre–Galerkin approximate solution of a Fredholm integro differential equation is obtained [10,27,26]. Let us call $e_n(x) = u(x) - u_n(x)$ as the error function of the Legendre approximation $u_n(x)$ to $u(x)$, where $u(x)$ is the exact solution of (1.1) and (1.2). Hence, $u_n(x)$ satisfies the following problem
\[
\sum_{i=0}^n \mu_i(x) u_n^{(i)}(x) - \lambda \int_{-1}^1 k(x, t) u_n(t) dt = f(x) + H_n(x),
\]
with boundary
\[
\sigma u_n(-1) = \sigma \sigma - 1 u_n(1) = 0,
\]
where $H_n$ is a perturbation term associated with $u_n(x)$ and can be obtained by substituting $u_n(x)$ into the equation
\[
H_n(x) = \sum_{i=0}^n \mu_i(x) u_n^{(i)}(x) - \lambda \int_{-1}^1 k(x, t) u_n(t) dt - f(x).
\]

We proceed to find an approximation $e_n(x)$ to the $e_n(x)$ in the same way as we did before for the solution (1.1) and (1.2). Subtracting (3.2) and (3.3) from (1.1) and (1.2), respectively, the error function $e_n(x)$ satisfies the equation
\[
\sum_{i=0}^n \mu_i(x) e_n^{(i)}(x) - \lambda \int_{-1}^1 k(x, t) e_n(t) dt = H_n(x),
\]
with boundary
\[
\sigma e_n(-1) = \sigma \sigma - 1 e_n(1) = 0.
\]
By solving this problem in the same way as Section 3, we get the approximation \( e_{n,N}(x) \). It should be noted that in order to construct the Legendre approximation \( e_{n,N}(x) \) to \( e_n(x) \).

4. The Legendre–Galerkin Method

We assume the solution of (1.1) is approximated by the finite expansion of Legendre basis function

\[
 u(x) = \sum_{j=0}^{n} c_j P_j(x). \tag{4.1}
\]

The unknown coefficients \( c_j \) in (4.1) are determined by orthogonalizing the residual with respect to the basis functions \( P_j(x) \). This yields the discrete system

\[
 (\mu_2(x)u''(x), P_r(x)) + (\mu_1(x)u'(x), P_r(x)) + (\mu_0(x)u(x), P_r(x)) = (f(x), P_r(x)) + \lambda \int_{-1}^{1} k(x,t)u(t)dt, \tag{4.2}
\]

The weighted inner product \( \langle \cdot, \cdot \rangle \) is taken to be

\[
 \langle \zeta, \eta \rangle = \int_{-1}^{1} \zeta(x) \cdot \eta(x) dx.
\]

The method of approximating the integrals in (4.2) begins by integrating by parts to transfer all derivatives from \( u \) to \( P_r \). Therefore, the following lemma is needed.

**Lemma 4.1.** The following relations hold

\[
 (\mu_1(x)u'(x), P_r(x)) = -\int_{-1}^{1} (\mu_1(x)P_r(x))' u(x) dx, \tag{4.3}
\]

\[
 (\mu_2(x)u''(x), P_r(x)) = [u'(x)\mu_2(x)P_r(x)]_{-1}^{1} + \int_{-1}^{1} (\mu_2(x)P_r(x))'' u(x) dx, \tag{4.4}
\]

\[
 (G(x), P_r(x)) \simeq \sum_{i=0}^{m} G(x_i) \frac{2}{(1-x_i^2)(P_m(x_i))^2}, \tag{4.5}
\]

\[
 \left\langle \int_{-1}^{1} F(x,t)dt, P_r(x) \right\rangle \simeq \sum_{i=0}^{m} \sum_{l=0}^{m} \omega_{ij} F(x_i,t_l) P_r(x_l), \tag{4.6}
\]

where

\[
 \omega_{ij} = \omega_i \omega_j = \frac{4}{(1-x_i^2)(1-x_j^2)(P_m(x_i)P_m(x_j))^2}.
\]

Replacing each term of (4.2) with the approximation defined in (4.3)–(4.6) respectively, we obtain the following theorem.

**Theorem 4.1.** If the assumed approximate solution of the boundary-value problem is (1.1), (1.2) is (4.1) then the discrete Galerkin–Legendre system for the determination of the unknown coefficients \( c_j \) is given by

\[
 \sum_{j=0}^{n} \left[ \sum_{i=0}^{m} \omega_i \omega_j k(x_i,t_l)P_j(x_l) - [u'(x)\mu_2(x)P_j(x)]_{-1}^{1} - \int_{-1}^{1} (\mu_2(x)P_j(x))'' P_r(x) dx + \int_{-1}^{1} (\mu_1(x)P_j(x))' P_r(x) dx \right] \cdot c_j = -\sum_{q=0}^{m} \omega_q f(x_q) P_r(x_q) dx. \tag{4.7}
\]

The system (4.7) takes the matrix form

\[
 Ac = b, \tag{4.8}
\]

where

\[
 A = \begin{bmatrix}
    \lambda h_{0,0} - e_{0,0} + v_{0,0} & \lambda h_{1,0} - e_{1,0} + v_{1,0} & \cdots & \lambda h_{n,0} - e_{n,0} + v_{n,0} \\
    \lambda h_{0,1} - e_{0,1} + v_{0,1} & \lambda h_{1,1} - e_{1,1} + v_{1,1} & \cdots & \lambda h_{n,1} - e_{n,1} + v_{n,1} \\
    \lambda h_{0,2} - e_{0,2} + v_{0,2} & \lambda h_{1,2} - e_{1,2} + v_{1,2} & \cdots & \lambda h_{n,2} - e_{n,2} + v_{n,2} \\
    \lambda h_{0,3} - e_{0,3} + v_{0,3} & \lambda h_{1,3} - e_{1,3} + v_{1,3} & \cdots & \lambda h_{n,3} - e_{n,3} + v_{n,3} \\
    \vdots & \vdots & \ddots & \vdots \\
    \lambda h_{0,n} - e_{0,n} + v_{0,n} & \lambda h_{1,n} - e_{1,n} + v_{1,n} & \cdots & \lambda h_{n,n} - e_{n,n} + v_{n,n} 
\end{bmatrix} \tag{4.9}
\]
and

\[
e_p = [\mu_2(x)P_j(x)P_r(x)]_{1-1}^{-1},
\]

\[
h_p = \sum_{i=0}^{m} \sum_{l=0}^{m} \alpha_i \omega_l k(x_i, t_l)P_j(t_l)P_r(x_i),
\]

\[
v_p = -\int_{-1}^{1} \mu_2(x)P_j'(x)P_r(x)dx - 2 \int_{-1}^{1} \mu_2(x)P_j'(x)P_j(x)dx - \int_{-1}^{1} \mu_2(x)P_r'(x)P_j(x)dx + \int_{-1}^{1} \mu_1(x)P_j'(x)P_j(x)dx + \int_{-1}^{1} \mu_1(x)P_j'(x)P_j(x)dx
\]

\[
v_p
\]

\[
\text{can be evaluated from theorems and lemmas in Section 2.}
\]

Now we have a linear system (4.8) of \(n + 1\) equations for \(n + 1\) unknown coefficients. We can obtain the coefficient of the approximate solution by solving this linear system by Q-R method. The solution \(c = (c_0, \ldots, c_n)^\top\) gives the coefficients in the approximate Legendre–Galerkin solution \(u(x)\).

\textbf{Algorithm}

- **INPUT**: \(n\) where \(n \in \mathbb{Z}\)
- if the domain is \([\alpha, \beta], x = \frac{\beta - \alpha}{2} s + \frac{\beta + \alpha}{2}\)
- evaluate \(A\) and \(b\)
- solve \(Ac = b\)
- **OUTPUT**: the values of \(c_0, c_1, c_2, \ldots, c_n\)
- \(s = \frac{\beta - \alpha}{2} x - \frac{\beta + \alpha}{2}\)
- end

\textbf{5. Treatment of boundary condition}

In Section 3 above, the development of Legendre–Galerkin technique for homogeneous boundary conditions, for \(n = 2\), if the boundary conditions are nonhomogeneous,

\[
u(-1) = \gamma, \quad u(1) = \varepsilon
\]

then these conditions need be converted to homogeneous conditions via an interpolation by a known function. Applying the transformation

\[
y(x) = u(x) - \frac{1 - x}{2} \gamma - \frac{x + 1}{2} \varepsilon,
\]

to the problem (1.1), (1.2) yields

\[
\sum_{i=0}^{2} \mu_i(x)y^{(i)}(x) = f(x) + \hat{\lambda} \int_{-1}^{1} k(x, t)y(t)dt
\]

with boundary conditions

\[
y(-1) = 0, \quad y(1) = 0,
\]

where

\[
\hat{f}(x) = f(x) - \frac{\varepsilon - \gamma}{2} \mu_1(x) - \left(\frac{(\varepsilon - \gamma)x + \varepsilon + \gamma}{2}\right) \mu_0(x) + \hat{\lambda} \int_{-1}^{1} k(x, t) \left(\frac{(t + 1)e + (1 - t)y}{2}\right)dt.
\]

The resulting discrete system for the coefficients \(n + 1\) in the approximate Legendre–Galerkin solution

\[
u(x) = \sum_{j=0}^{n} c_j P_j(x) + \frac{1 - x}{2} \gamma + \frac{x + 1}{2} \varepsilon
\]

is exactly the system in (4.8), with \(f\) in that system replaced by \(\hat{f}\). Notice that if \(\gamma = \varepsilon = 0\), the problem reduces to the homogeneous case.
6. Numerical examples

In this section, we apply Legendre–Galerkin method to various problems which were collected from available literatures [7,11,20,18,15,17]. Our primary interest is to compare the method on the same problems. All computations were carried out using Matlab and Mathematica on a personal computer.

In the following examples, the maximum absolute error is taken as

\[ \|E_{LG}\| = |u_{exact}(x) - u_{Legendre-Galerkin}(x)| \]

**Example 1** ([7,11,20,18]). Consider the integro-differential equation

\[
u'(x) - \nu(x) = -\cos(2\pi x) - 2\pi \sin(2\pi x) - \frac{1}{2} \sin(4\pi x) + \int_0^1 \sin(4\pi x + 2\pi t)u(t)dt, \quad 0 < x < 1,
\]

subject to the initial condition

\[ u(0) = 1 \]

whose exact solution is

\[ u(x) = \cos(2\pi x). \]

Maximum absolute error are tabulated in Table 1 at different \( n \). Table 2 exhibits a comparison between the errors obtained by using the Legendre–Galerkin, using the homotopy analysis method [11], using the Haar wavelet method [20], using CAS wavelet [7], and using sinc basis functions [18] beside the estimate error \( e_n \) for Legendre–Galerkin method. Fig. 1(a) shows the maximum absolute error at different \( n \) and Fig. 1(b) presents the Legendre and exact solutions.

**Example 2** ([15,17]). Consider the Fredholm integral equation

\[
u(x) = e^x - \frac{e^{x^2 + 1}}{x + 1} + \int_0^1 e^{x^2}u(t)dt, \quad 0 < x < 1,
\]

whose exact solution is

\[ u(x) = e^x. \]

The numerical results are displayed in Table 3 for different values of \( n \). Maximum absolute error is tabulated in Table 4 for Legendre–Galerkin together with the numerical results of Mohsen and El-Gamel [17], who used the sinc-collocation method to obtain the numerical solution and Maleknejad and Mirzaee [15] who used the Haar wavelet method to obtain the approximate solution. Fig. 2(a) introduces the maximum absolute error at different \( n \) and Fig. 2(b) exhibits the Legendre and exact solutions.

**Example 3.** Consider the integro-differential equation

\[ x^2u''(x) - xu'(x) + u(x) = -\pi x \cos(\pi x) - \pi^2 x^2 \sin(\pi x) + \int_{-1}^1 \cos(\pi x - \pi t) u(t)dt, \quad -1 < x < 1. \]
Fig. 1. (a) Comparison of $|E_n|$ with $n = 5, 10, 15, 20, 25$ for example 1. (b) Legendre and exact solutions for example 1.

Table 3
Maximum absolute errors and maximum estimate error for Example 2 at different $n$.

| $N$ | Legendre–Galerkin $|E_n|$ | $|e_n|$ |
|-----|-------------------------|--------|
| 5   | 5.764E–05               | 8.700E–06 |
| 10  | 2.530E–12               | 1.124E–12 |
| 15  | 1.776E–15               | 2.288E–15 |
| 20  | 1.332E–15               | 2.170E–15 |

Fig. 2. (a) Comparison of $|E_n|$ with $n = 5, 10, 15, 20$ for example 2. (b) Legendre and exact solutions for example 2.

Table 4
Comparison of maximum absolute errors for example 2.

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<tr>
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<tbody>
<tr>
<td>2.442E–15</td>
<td>0.4756 E–06</td>
<td>0.0413</td>
</tr>
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</table>
subject to the boundary conditions

\[ u(-1) = u(1) = 0, \]

whose exact solution is

\[ u(x) = \sin(\pi x). \]

The computational results are summarized in Table 5, for different values of \( n \). Fig. 3 (a) displays the maximum absolute error and maximum estimate error at different \( n \) and Fig. 3 (b) bids the Legendre and exact solutions.

**Example 4** [23]. Consider the integro-differential equation

\[ u'' + xu' - xu = e^x - 2 \sin x + \sin x \int_{-1}^{t} e^{-t} u(t) \, dt. \quad -1 < x, t < 1, \]

subject to the boundary conditions

\[ u(-1) = e^{-1}, \quad u(1) = e, \]

whose exact solution is

\[ u(x) = e^x. \]

The computational results are summarized in Table 6 for different values of \( n \). Maximum absolute error is tabulated in Table 7 for Legendre–Galerkin together with the numerical results of S. Yalçınbaş [23], who used Legendre-collocation matrix

| \( N \) | Legendre–Galerkin \( ||E_{\text{GC}}|| \) | \( \|e_{n}\| \) |
|---|---|---|
| 5 | 6.006E−01 | 2.919E−01 |
| 10 | 1.008E−04 | 8.528E−05 |
| 15 | 3.282E−08 | 2.936E−08 |
| 20 | 3.752E−14 | 4.244E−14 |

![Fig. 3. (a) Comparison of \( ||E_{\text{GC}}|| \) with \( n = 5, 10, 15, 20 \) for example 3. (b) Legendre and exact solutions for example 3.](image)

**Table 6**

Maximum absolute errors and maximum estimate error for Example 4 at different \( n \).

| \( N \) | Legendre–Galerkin \( ||E_{\text{GC}}|| \) | \( \|e_{n}\| \) |
|---|---|---|
| 5 | 2.136E−03 | 6.324E−03 |
| 10 | 1.008E−09 | 9.187E−10 |
| 15 | 8.104E−15 | 1.297E−15 |
method to obtain the numerical solution. Fig. 4(a) shows the maximum absolute error at different $n$ and Fig. 4(b) exhibits the Legendre and exact solutions.

7. Conclusion

This paper has discussed how the Legendre–Galerkin method can be applied for obtaining solutions of integral and integro-differential equations. The formulation and implementation of the scheme are illustrated. The proposed method was tested using some problems with results. This Paper discussed how the integro-differential equation with variable coefficients can be solved using the Legendre–Galerkin method. Matlab and Mathematica had been used to obtain the approximate solution. If the problem domain is $[a, b/C_{138}]$, the linear transformation must be used to convert the problem domain to $[1/C_{01}, 1/C_{138}]$. The required linear transformation is $s = \frac{b-x}{a-x}$. Then, by using Legendre–Galerkin technique, the approximate solution is converted to linear algebraic system. By solving this system, the numerical solution is obtained. Finally, By using inverse transformation, the final approximate solution can be produced.

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References


