FUZZY INTEGRALS AND CONDITIONAL FUZZY MEASURES

Rudolf KRUSE

Institut für Angewandte Mathematik der TU Braunschweig, Braunschweig, Federal Republic of Germany

Received July 1982
Revised August 1982

We introduce a fuzzy integral for fuzzy events with respect to λ-additive fuzzy measures. This integral is the canonical generalization of the Lebesgue integral. A Radon-Nikodym-like theorem is used to give the definition of the conditional fuzziness.

Keywords: Fuzzy measure, Fuzzy integral.

1. Introduction

The concept of fuzzy sets suggested by Zadeh [9] gave us a powerful tool by which we can treat many complicated problems of human behaviour. Sugeno [6] considered a second view of fuzziness: the one of certainty degrees for assertions such as ‘u ∈ A’ where A is a nonfuzzy set and u is a variable whose value is not perfectly located in a nonfuzzy set X, with respect to A. Sugeno’s approach generalizes probability measures by dropping the additivity property and replacing a weaker one, i.e. monotonicity. The specification of a fuzzy measure requires the knowledge of g(A) for all subsets A in X. In order to reduce the quantity of primary data, Sugeno [7] added a further axiom, i.e. λ-additivity. More details concerning these special fuzzy measures can be found in a paper by Banon [1]. Kruse [3] showed that there exists a relationship between probability measures and λ-additive fuzzy measures. This relationship is used to give the definition of a so called ‘fuzzy integral’ of a fuzzy event with respect to a λ-additive fuzzy measure, which generalizes the Lebesgue integral canonically. For λ-additive fuzzy measures our integral is the proper tool to express fuzzy expectations.

In Section 3 a Radon-Nikodym-like theorem for λ-additive fuzzy measures is given. This theorem is used to formulate the general definition of the conditional fuzziness.

2. Fuzzy integrals

Let X be an arbitrary set and let Σ be a σ-algebra on X. A function g from Σ to
[0, 1] is said to be a fuzzy measure [6], if and only if the following conditions hold:

\[ g(\emptyset) = 0, \quad g(\mathcal{X}) = 1, \quad (2.1) \]

if \( A, B \in \mathcal{U} \) and \( A \subseteq B \), then \( g(A) \leq g(B) \),

\[ (2.2) \]

if \( (A_n) \) is a monotone sequence of sets in \( \mathcal{U} \), then

\[ \lim_{n \to \infty} g(A_n) = g\left( \lim_{n \to \infty} A_n \right). \quad (2.3) \]

Let \( \lambda \in (-1, \infty) \), \( \lambda \neq 0 \), be a real number. A fuzzy measure \( g \) on \( \mathcal{U} \) is called \( \lambda \)-additive [7] if, whenever

\[ A \in \mathcal{U}, \quad B \in \mathcal{U}, \quad \text{and} \quad A \cap B = \emptyset, \]

then

\[ g(A \cup B) = g(A) + g(B) + \lambda g(A)g(B). \]

If \( g \) is a \( \lambda \)-additive fuzzy measure on \( \mathcal{U} \), then

\[ g^*: = \frac{[\log(1 + \lambda g)]}{[\log(1 + \lambda)]} \]

is a probability measure on \( \mathcal{U} \) [3]. Thus we can use the \( g^* \)-integral to define a \( g \)-integral.

**Definition 1.** Let \( g \) be a \( \lambda \)-additive fuzzy measure on \( \mathcal{U} \), and let \( f: X \to [0, 1] \) be a \( \mathcal{U} \)-measurable function. \( f \) is then called a fuzzy event [10]; we write \( f \in F_c(\mathcal{U}) \). For an arbitrary \( A \in \mathcal{U} \) we define

\[ \int_A f \,dg := \frac{1}{\lambda} + \frac{1}{\lambda} (1 + \lambda)^{\log(1 + \lambda) / \log(1 + \lambda) + 1}, \]

and name the quantity the \( g \)-integral of \( f \) over \( A \).

**Example 1.** Let \( \mu \) be the restriction of the Lebesgue measure on \( \mathcal{U} \cap [0, 1) \). Then

\[ \chi_\lambda := -\frac{1}{\lambda} + \frac{1}{\lambda} (1 + \lambda)^{\mu} \]

is the unique \( \lambda \)-additive fuzzy measure \( h \) on \( \mathcal{U} \cap [0, 1) \) such that \( h \) is invariant against translations [2]. Let \( f: [0, 1) \to [0, 1] \),

\[ f(t) := \sum_{i=1}^n \gamma_i \cdot I_{(i-1/n, i/n)}(t), \quad \text{where} \quad \gamma_i \in [0, 1] \]

\((I_A \) is the indicator function of \( A \)). Then we have

\[ \int_{[0,1)} f \,d\chi_\lambda = \frac{1}{\lambda} + \frac{1}{\lambda} (1 + \lambda)^{\log(1 + \lambda) / \log(1 + \lambda) + 1}, \]

\[ = \frac{1}{\lambda} + \frac{1}{\lambda} (1 + \lambda)^{\log(1 + \lambda) / \log(1 + \lambda) + 1}, \]

\[ = \frac{1}{\lambda} + \frac{1}{\lambda} (1 + \lambda)^{\log(1 + \lambda) / \log(1 + \lambda) + 1}, \]

\[ = \frac{1}{\lambda} \left[ \prod_{i=1}^n (1 + \lambda\gamma_i)^{1/n} - 1 \right]. \]
We can interpret this quantity as a kind of weighted mean of the $x_i$: it follows
\[
\lim_{h \to 0} \int_{[0,1]} f \, dg_h = \frac{1}{n} \sum_{i=1}^{n} x_i \quad \text{(arithmetic mean)},
\]
\[
\lim_{h \to 1} \int_{[0,1]} f \, dg_h = 1 - \prod_{i=1}^{n} (1 - x_i)^{1/n}.
\]
\[
\lim_{h \to \infty} \int_{[0,1]} f \, dg_h = \prod_{i=1}^{n} x_i^{1/n} \quad \text{(geometric mean)}.
\]

The proof of the following theorem follows from the fact that the function
\[
h(t) := -\frac{1}{t} + \frac{1}{\lambda t} (1 + \lambda t).
\]
is continuous and increasing, where $h(0) = 0$, $h(1) = 1$.

**Theorem 1.** The set function $h$, $h(A) = \int f \, dg$, is a $\lambda$-additive fuzzy measure on $\mathcal{L}$, if and only if
\[
\int_{\mathcal{L}} \log(1 + \lambda f) \, dg = 1.
\]

### 3. Conditional $\lambda$-additive fuzzy measures

Let $g$ be a $\lambda$-additive fuzzy measure on $\mathcal{L}$ and let $A \in \mathcal{L}$ with $g(A) > 0$. For all $B \in \mathcal{L}$ we define
\[
g_A(B) := \frac{1}{\lambda} + \frac{1}{\lambda} (1 - \lambda) \log(1 + \lambda) \log(1 + \lambda A \setminus B).
\]
and name the quantity the conditional fuzziness of $B$ given $A$. Then $g_A$, where $g_A(B) := g(B \mid A)$ for all $B \in \mathcal{L}$, is a $\lambda$-additive fuzzy measure on $\mathcal{L}$ (see [5]).

A Radon–Nikodym-like theorem is used to give the general definition of the conditional fuzziness.

**Theorem 2.** Let $g$ be a $\lambda$-additive fuzzy measure on $\mathcal{L}$, and let $h : \mathcal{L} \to [0, 1]$ be a $\lambda$-additive and continuous (2.3) function such that
\[
\text{if } N \in \mathcal{L} \text{ and } g(N) = 0, \text{ then } h(N) = 0.
\]

Then there exists a function $f \in F_\lambda(\mathcal{L})$ such that
\[
h(B) = \int_B f \, dg \quad \text{for every } B \in \mathcal{L}.
\]

The function is unique in the sense that if $h(B) = \int_B f' \, dg$, $B \in \mathcal{L}$, then $f = f'$ $g$-almost everywhere.
Proof. We define $g^*$, $h^*$ as in (2.4), and we know that $g^*$ is a probability measure. It is easy to show that $h^*$ is a (usual) measure on $\mathcal{F}$. (3.1) gives that $h^*$ is absolutely continuous with respect to $g^*$. From the Radon–Nikodym theorem we know that there exists a function $f^* \in F_c(\mathcal{F})$ such that $h^*(B) = \int_B f^* \, dg^*$ for every $B \in \mathcal{F}$. We define

$$f := -\frac{1}{\lambda} + \frac{1}{\lambda} (1 + \lambda) f^*.$$ 

and conclude $h(B) = \int_B f \, dg$ for every $B \in \mathcal{F}$. If $h(B) = \int_B f' \, dg$, $B \in \mathcal{F}$, then

$$h^*(B) = \int_B \frac{\log(1 + \Lambda f')}{\log(1 + \lambda)} \, dg^*, \quad B \in \mathcal{F},$$

thus

$$\frac{\log(1 + \Lambda f')}{\log(1 + \lambda)} = f^*$$
g-almost everywhere. Therefore we have $f' = f$ g-almost everywhere.

Theorem 3. If $g$ is a $\lambda$-additive fuzzy measure on $\mathcal{F}$, if $\mathcal{M}$ is a $\sigma$-subalgebra of $\mathcal{F}$, and if $B \in \mathcal{F}$, then there exists a function $f \in F_c(\mathcal{M})$ such that

$$g(A \cap B) = \int_A f \, dg \quad \text{for every } A \in \mathcal{M}.$$ 

$f$ is unique g-almost everywhere.

Proof. $h : \mathcal{M} \to [0, 1]$, $h(A) = g(A \cap B)$, is $\lambda$-additive and continuous (see Theorem 1), and we have $0 \leq h(A) \leq g(A)$ for all $A \in \mathcal{M}$. Theorem 2 gives the proof.

Definition 2. Given the situation in Theorem 3 we call the function $f$ the conditional fuzziness of $B$ under the condition $\mathcal{M}$, we write $g(B \mid \mathcal{M}) := f$.

Example 2. Let $t : X \to [0, 1]$ be a $\mathcal{F}$-measurable function such that $t(X) = \{t_1, \ldots, t_n\}$, $A_k := \{x : t(x) = t_k\}$, $g(A_k) > 0$, $k = 1, 2, \ldots, n$, $n = 1, 2, 3, \ldots$, and let $B \in \mathcal{F}$. We define $g(B \mid t) := g(B \mid t^{-1}(\{t^1\}))$. Then we have

$$g(B \mid t) \in F_c(t^{-1}(\{t^1\})), \quad g(B \mid t) = \sum_{k=1}^n g(B \mid A_k) I_{A_k}.$$ 

References
