INTRODUCTION

ARTIFICIAL VARIABLE TECHNIQUE

THE INTRODUCTION OF SLACK/SURPLUS VARIABLES PROVIDED THE INITIAL BASIC FEASIBLE SOLUTION. BUT THERE ARE MANY PROBLEMS WHEREIN AT LEAST ONE OF THE CONSTRAINTS IS OF ($\geq$) OR (=) TYPE AND SLACK VARIABLES FAIL TO GIVE SUCH A SOLUTION.

M-METHOD - DUE TO A. CHARNES.

STEP 1 : EXPRESS THE PROBLEM IN STANDARD FORM

STEP 2 : ADD NON-NEGATIVE VARIABLES (ARTIFICIAL VAR.) TO THE LEFT-HAND SIDE OF ALL THOSE CONSTRAINTS WHICH ARE OF ($\geq$) OR (=) TYPE.
ASSIGN A VERY LARGE PENALTY (−M) TO THESE ARTIFICIAL VAR. IN THE OBJ. FUNCTION.

STEP 3 : SOLVE THE MODIFIED LPP BY SIMPLEX METHOD.

EXAMPLE

USE CHARNE'S PENALTY METHOD TO

MINIMIZE $Z = 2x_1 + x_2$

SUBJECT TO $3x_1 + x_2 = 3$

$4x_1 + 3x_2 \geq 6$

$x_1 + 2x_2 \leq 3$

$x_1, x_2 \geq 0$

SOLUTION

STEP 1: EXPRESS THE PROBLEM IN STANDARD FORM

THE SECOND AND THIRD INEQUILITIES ARE CONVERTED INTO EQUATIONS BY INTRODUCING THE SURPLUS AND SLACK VARIABLES S_1, S_2 RESPECTIVELY.

THE FIRST AND SECOND CONSTRAINTS BEING OF (=) AND ($\geq$) TYPE, INTRODUCE 2 ARTIFICIAL VARIABLES A_1, A_2.

CONVERTING THE MINIMIZATION PROBLEM TO THE MAXIMIZATION FORM

$$\text{MAX} \quad Z' = -2x_1 - x_2 + 0s_1 + 0s_2 - MA_1 - MA_2.$$

SUBJECT TO
$$3x_1 + x_2 + 0s_1 + 0s_2 + A_1 + 0A_2 = 3$$
$$4x_1 + 3x_2 - s_1 + 0s_2 + 0A_1 + A_2 = 6$$
$$x_1 + 2x_2 + 0s_1 + s_2 + 0A_1 + 0A_2 = 3$$
$$x_1, x_2, s_1, s_2, A_1, A_2 \geq 0$$

STEP 2: OBTAIN AN INITIAL FEASIBLE SOLUTION

$x_1 = x_2 = 0$; $s_1 = 0$;

$A_1 = 3$, $A_2 = 6$, $S_2 = 3$.

	C_j	-2	-1	0	0	$-M$	$-M$		
CB	Basis	x_1	x_2	s_3	s_3	A_1	A_2	b	θ
$-M$	A_1	(3)	1	0	0	1	0	3	3/3 ←
$-M$	A_2	4	3	-1	0	0	1	6	6/4
0	S_2	1	2	0	1	0	0	3	3/1
	Z_j	$-7M$	$-4M$	M	0	$-M$	$-M$	$-9M$	
	C_j	$7M-2$ ↑	$4M-1$	$-M$	0	0	0		

SINCE C_j IS POSITIVE UNDER x_1 AND x_2 COLUMNS,

THIS IS NOT AN OPTIMAL SOLUTION.

STEP 3 : ITERATE TOWARDS OPTIMAL SOLUTION

INTRODUCE x_1 AND DROP A_1

C_B	c_j / Basis	-2 / x_1	-1 / x_2	0 / S_1	0 / S_2	$-M$ / A_2	b	θ
-2	x_1	1	$1/3$	0	0	0	1	3
$-M$	A_2	0	$(5/3)$	-1	0	1	2	$6/5$ ←
0	S_2	0	$5/3$	0	1	0	2	$6/5$
	Z_j	-2	$-\frac{2}{3}-\frac{5M}{3}$	M	0	$-M$	$-2-2M$	
	C_j	0	$-\frac{1}{3}+\frac{5M}{3}$ ↑	$-M$	0	0		

SINCE C_j IS POSITIVE UNDER x_2 COLUMN, THIS IS NOT AN OPTIMAL SOLUTION

INTRODUCE x_2 AND DROP A_2

	C_j	-2	-1	0	0	
C_B	Basis	x_1	x_2	s_1	s_2	b
-2	x_1	1	0	$1/5$	0	$3/5$
-1	x_2	0	1	$-3/5$	0	$6/5$
0	s_2	0	0	1	1	0
	Z_j	-2	-1	$4/5$	0	$-12/5$
	C_j	0	0	$-1/5$	0	

SINCE NONE OF C_j IS POSITIVE, THIS IS AN OPTIMAL SOLUTION. THUS AN OPTIMAL BASIC FEASIBLE SOLUTION TO THE PROBLEM IS

$$x_1 = 3/5 \quad , \quad x_2 = 6/5 \quad , \quad Z_{MAX} = -12/5$$

HENCE THE OPTIMAL VALUE OF THE OBJ. FUCTION IS $\min Z = -\max Z'$

$$= -(-12/5)$$

$$= 12/5$$

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EXAMPLE

MAXIMIZE $Z = 3x_1 + 2x_2$

SUBJECT TO
$$2x_1 + x_2 \leq 2$$
$$3x_1 + 4x_2 \geq 12$$
$$x_1, x_2 \geq 0$$

SOLUTION

STEP 1 EXPRESS THE PROBLEM IN STANDARD FORM

INTRODUCE SLACK/SURPLUS VARIABLES

2ND CONSTRAINT ($\geq$), INTRODUCE ARTIFICIAL VAR. A

MAX $Z = 3x_1 + 2x_2 + 0S_1 + 0S_2 - MA$

SUBJECT TO
$$2x_1 + x_2 + S_1 + 0S_2 + 0A = 2$$
$$3x_1 + 4x_2 + 0S_1 - S_2 + A = 12$$
$$x_1, x_2, S_1, S_2, A \geq 0$$

STEP 2 OBTAIN AN INITIAL BASIC FEASIBLE SOLUTION

$$x_1 = x_2 = S_2 = 0$$
$$S_1 = 2, \quad A = 12$$

	C_j	3	2	0	0	$-M$		
C_B	Basis	x_1	x_2	s_1	s_2	A	b	θ
0	s_1	2	(1)	1	0	0	2	2
$-M$	A	3	4	0	-1	1	12	3
	Z_j	$-3M$	$-4M$	0	M	$-M$	$-12M$	
	C_j	$3+3M$	$2+4M$	0	$-M$	0		

why not s_2?

SINCE C_j IS POSITIVE UNDER SOME COLUMN ∄ NOT OPTIMAL SOLUTION

STEP 3 ITERATE TOWARDS OPTIMAL SOLUTION

INTRODUCE x_2 AND DROP s_1

	C_j	3	2	0	0	$-M$	
C_B	Basis	x_1	x_2	s_1	s_2	A	b
2	x_2	2	1	1	0	0	2
$-M$	A	-5	0	-4	-1	1	4
	Z_j	$4+5M$	2	$2+4M$	M	$-M$	$4-4M$
	C_j	$-(1+5M)$	0	$-(2+4M)$	$-M$	0	

C_j IS NEGATIVE AND AN ARTIFICIAL VAR. APPEARS IN THE BASIS AT NON-ZERO LEVEL. THUS THERE EXISTS A PSEUDO OPTIMAL SOLUTION TO THE PROBLEM.

MORE EXAMPLES

Big-M method

1) Minimize $z = 4x_1 + x_2$

Subject to :

i) $3x_1 + x_2 = 3$
ii) $4x_1 + 3x_2 \geq 6$
iii) $x_1 + 2x_2 \leq 4$

$x_1, x_2 \geq 0$

Solution :

Std form of the equation;

Max $z' = -4x_1 - x_2 + 0S_1 + 0S_2 - MA_1 - MA_2$

i) $3x_1 + x_2 + 0S_1 + 0S_2 + A_1 + A_2 = 3$
ii) $4x_1 + 3x_2 - 0S_1 + 0S_2 + 0A_1 + A_2 = 6$
iii) $x_1 + 2x_2 + 0S_1 + S_2 + 0A_1 + 0A_2 = 4$

$x_1, x_2, S_1, S_2, A_1, A_2 \geq 0$

	Cj	-4	-1	0	0	-M	-M		
cb	Basic	x_1	x_2	S_1	S_2	A_1	A_2	b	θ
-M	A_1	3	1	0	0	1	0	3	1
-M	A_2	4	3	-1	0	0	1	6	6/4
0	S_2	1	2	0	1	0	0	4	4
	zj	-7M	-4M	M	0	-M	-M	-9M	
	Cj	-4+7M	-1+4M	-M	0	0	0		

	Cj	-4	-1	0	0	-M		
cb	Basic	x_1	x_2	S_1	S_2	A_2	b	θ
-4	x_1	1	1/3	0	0	0	1	3
-M	A_2	0	5/3	-1	0	1	2	6/5
0	S_2	0	5/3	0	1	0	3	9/5
	zj	-4	-4/3 -5/3M	M	0	-M	-4 -2M	
	Cj	0	1/3 + 5/3M	-M	0	0		

	Cj	-4	-1	0	0		
cb	Basic	x_1	x_2	S_1	S_2	b	θ
-4	x_1	1	0	1/5	0	3/5	
-1	x_2	0	1	-3/5	0	6/5	
0	S_2	0	0	0	1	1	
	zj	-4	-1	-1/5	0	-18/5	
	Cj	0	0	1/5	0		

since ~~zj~~ cj is negative, stop the iteration

min z = - (- 18/5)

= 18/5

x_1 = 3/5 and x_2 = 6/5

2) a) **Introduction**

The introduction of slack/surplus variables provided the initial basic feasible solution. But there are many problems where in at least one of the constraints is of ($\geq$) or ($=$) type and slack variables fail to give such a solution.

b) **Procedure**

- Step 1: Express the problem in standard form
- Step 2: Add non-negative variables (Artificial variables) to the left-hand side of all those constraints which are of ($\geq$) or ($=$) type. Assign a very large penalty ($-M$) to these artificial variable in the objective function.
- Step 3: Solve the modified LPP by simplex method.

Question:

Min. $Z = 2x_1 + 3x_2$

Subject to
$$1/2\, x_1 + 1/4\, x_2 \leq 4$$
$$x_1 + 3x_2 \geq 20$$
$$x_1 + x_2 = 10$$
$$x_1, x_2 \geq 0$$

Solution:

i. Let S_1, S_2, A_1, A_2 be the three slack variables.

Modified form is:

Max $Z = -2x_1 - 3x_2 + 0S_1 + 0S_2 - MA_1 - MA_2$

Subject to
$$1/2\, x_1 + 1/4\, x_2 + S_1 + 0S_2 + 0A_1 + 0A_2 = 4$$
$$x_1 + 3x_2 + 0S_1 - S_2 + A_1 + 0A_2 = 20$$
$$x_1 + x_2 + 0S_1 + 0S_2 + 0A_1 + A_2 = 10$$
$$x_1, x_2, S_1, S_2, A_1, A_2 \geq 0$$

ii. Initial feasible solution.

$$x_1 = 0,\ x_2 = 0,$$
$$S_1 = 4,\ A_1 = 20,\ A_2 = 10$$

	c_j	-2	-3	0	0	$-M$	$-M$		
C_B	*Basis*	x_1	x_2	S_1	S_2	A_1	A_2	b	θ
0	S_1	$1/2$	$1/4$	1	0	0	0	4	16
$-M$	A_1	1	3	0	-1	1	0	20	$20/3$
$-M$	A_2	1	1	0	0	0	1	10	10
	Z_j	$-2M$	$-4M$	0	M	$-M$	$-M$	$-30M$	
	C_j	$-2+2M$	$-3+4M$	0	$-M$	0	0		

iii. Introduce x_2, drop A_1

	c_j	-2	-3	0	0	$-M$		
C_B	*Basis*	x_1	x_2	S_1	S_2	A_2	b	θ
0	S_1	$5/12$	0	1	$1/12$	0	$7/3$	$28/5$
-3	x_2	$1/3$	1	0	$-1/3$	0	$20/3$	20
$-M$	A_2	$2/3$	0	0	$1/3$	1	$10/3$	5
	Z_j	$-1-\frac{2}{3}M$	-3	0	$1-\frac{M}{3}$	$-M$	$-20-\frac{10}{3}M$	
	C_j	$-1+\frac{2}{3}M$	0	0	$-1+\frac{M}{3}$	0		

iv. Introduce x_1, drop A_2

	c_j	-2	-3	0	0	
C_B	*Basis*	x_1	x_2	S_1	S_2	b
0	S_1	0	0	1	$-1/8$	$1/4$
-3	x_2	0	1	0	$-1/2$	5
-2	x_1	1	0	0	$1/2$	5
	Z_j	-2	-3	0	-1	-25
	C_j	0	0	0	1	

- Optimal Solution is : $x_1 = 5$

  $x_2 = 5$

  $Z_{\max} = -25$, $Z_{\min} = 25$

3) Maximize $z = x_1 + x_2$

Subject to

$$2x_1 + x_2 \geq 4$$

$$x_1 + 2x_2 = 6$$

$$x_1, x_2 \geq 0$$

Solution:

Step 2: Express the problem in the standard form. Both inequalities are converted into equalities by introducing the surplus and slack variables S_1, S_2 respectively. Artificial variables A_1 and A_2 are being introduced too.

The problem in standard form becomes

$$\text{Max } z = x_1 + x_2 + 0s_1 + 0s_2 - MA_1 - MA_2$$

Subject to

$$2x_1 + x_2 - s_1 + 0s_2 + A_1 = 4$$
$$x_1 + 2x_2 + 0s_1 + 0s_2 + A_2 = 6$$

$$x_1, x_2, s_1, s_2, A_1, A_2 \geq 0$$

Step 3: Find initial basic feasible solution. The basic feasible solution is

$$x_1 = x_2 = s_1 = s_2 = 0 \ (non - basic) \quad A_1 = 4, A_2 = 6 (basic)$$

	Cj	1	2	0	0	-M	-M		
CB	basis	x_1	x_2	S_1	S_2	A_1	A_2	b	Ø
-M	A_1	2	1	-1	0	1	0	4	4/1=4
-M	A_2	1	(2)	0	0	0	1	6	6/2=3
	Zj	-3M	-3M	M	0	-M	-M	-10M	
	Cj	1+3M	2+3M	-M	0	0			

Step 4: Apply optimality test as Cj is positive under first column, the initial feasible solution is not optimal.

Step 5: Identify the incoming and outgoing variables

X_2 is the incoming variable

A2 is the outgoing variable

(2) is the key element

Iterate towards the optimal solution by introducing X_2 and dropping A2,

Convert the key element to unity and make other element of the key column to zero.

	Cj	1	2	0	0	-M		
CB	basis	x_1	x_2	S_1	S_2	A_1	b	Ø
-M	A_1	(3/2)	0	-1	0	1	1	1/3/2=2/3
2	x_2	1/2	1	0	0	0	3	3/1/2=6
	Zj	1-3/2M	2	M	0	-M	6-M	
	Cj	3/2M	0	-M	0	0		

R2 = r2 – r1

Step 6 : as Cj is positive under first column, the solution is not optimal. Here X_1 is the incoming variable and A_1 is the outgoing variable and (0.5) is the key element.

Introduce X_1 and drop A_1

Concert the key element to unity

Make all other elements of the key column zero

	Cj	1	2	0	0	-M		
C_B	basis	X_1	X_2	S_1	S_2	A_1	b	Ø
1	X_1	1	0	-2/3	0	2/3	2/3	
2	X_2	0	1	1/3	0	-1/3	8/3	
	Zj	1	2	0	0	0	18/3	
	Cj	0	0	0	0	-M		

R2 = r2 −1/2r1

Since Cj ≤ 0, therefore the optimal feasible solution is

$X_1 = 2/3$, $X_2 = 8/3$, Z max = 18/3

Solve the following LPP using M- method

1. Max $Z = 3x_1 + 2x_2 + 3x_3$

Subject to $2x_1 + x_2 + x_3 \leq 2$

$3x_1 + 4x_2 + 2x_3 \geq 8$

$x_1, x_2, x_3 \geq 0$

2. Maximize $Z = 2x_1 + x_2 + 3x_3$

Subject to $x_1 + x_2 + 2x_3 \leq 5$

$2x_1 + 3x_2 + 4x_3 = 12$

$x_1, x_2, x_3 \geq 0$

3. ~~Maximize~~ $\cancel{Z = 8x_2}$

Minimize $Z = 4x_1 + 3x_2 + x_3$

Subject to $x_1 + 2x_3 + 4x_3 \geq 12$

$3x_1 + 2x_2 + x_3 \geq 8$

$x_1, x_2, x_3 \geq 0$